Cellularity, composition, and morphisms of algebraic weak factorization systems

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Algebraic weak factorization systems

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- has left and right classes $\mathcal{L}$ and $\mathcal{R}$ of maps s.t. $$\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ {\scriptstyle \mathcal{L}\ni\ell}\downarrow & & \downarrow{\scriptstyle r\in\mathcal{R}} \\ \cdot & \longrightarrow & \cdot \end{array}$$

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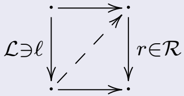

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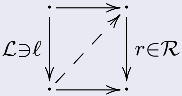

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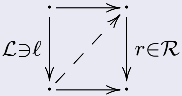

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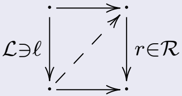

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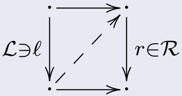

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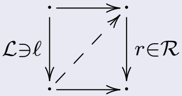

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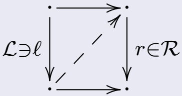

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- (injective with projective cokernel, surjective) in $\mathbf{Mod}_R$

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Motivating example

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- Hence, we call the maps admitting a coalgebra structure cellular.
- Not all cofibrations (elements of the left class of the weak factorization system) are cellular: cellularity is a condition!
- Generic cofibrations are retracts of relative cell complexes, equivalently, coalgebras for the pointed endofunctor of the comonad.

Composition

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In general

- Coalgebras for the comonad of an algebraic weak factorization system can be composed and the composition is functorial.

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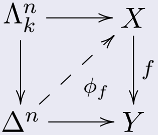

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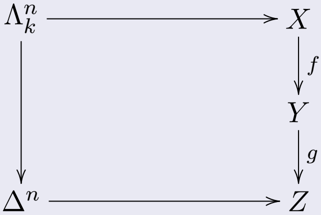

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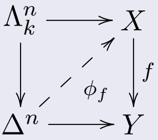

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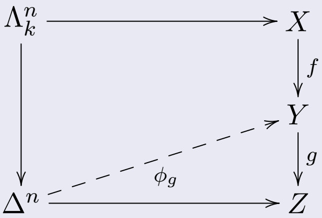

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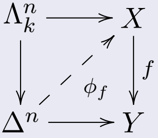

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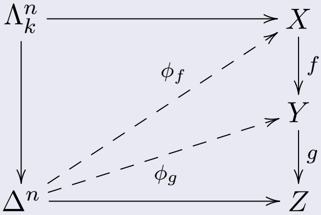

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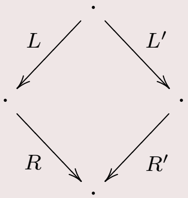
Preliminary definition.

A morphism between two algebraic weak factorization systems is

- a natural transformation comparing their functorial factorizations



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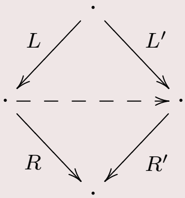

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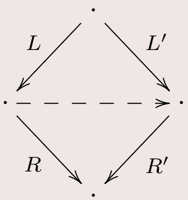

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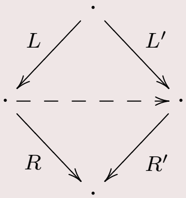

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We will define morphisms between algebraic weak factorization systems on different categories lifting (two-variable) adjunctions.

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$$\begin{array}{ccc} \cdot & \xrightarrow{u} & \cdot \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \cdot & \xrightarrow[v]{} & \cdot \end{array} \quad \mapsto \quad \begin{array}{ccc} \cdot & \xrightarrow{u} & \cdot \\ {\scriptstyle Lf}\downarrow & & \downarrow{\scriptstyle Lg} \\ \cdot & \longrightarrow & \cdot \\ {\scriptstyle Rf}\downarrow & & \downarrow{\scriptstyle Rg} \\ \cdot & \xrightarrow[v]{} & \cdot \end{array} \quad \text{with } Lf \in \mathcal{L}, Rf \in \mathcal{R}.$$

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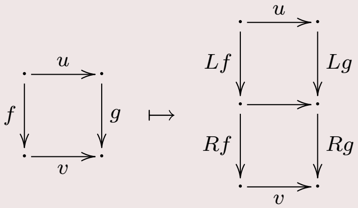

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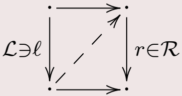

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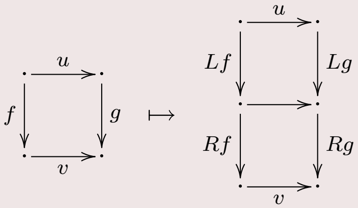

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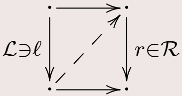

- (closure) furthermore $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$

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Left maps are coalgebras and right maps are algebras, resp., for the pointed endofunctors $L, R\colon \mathcal{M}^{\mathbf{2}} \rightrightarrows \mathcal{M}^{\mathbf{2}}$ with $\epsilon\colon L \Rightarrow 1$, $\eta\colon 1 \Rightarrow R$.

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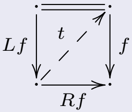

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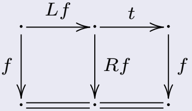

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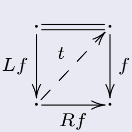

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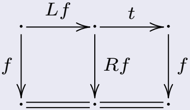

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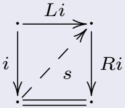

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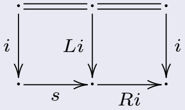

iff $i \in (L, \epsilon)$-coalg

Algebraic lifts

Recall

$$i \in \mathcal{L} \quad \text{iff} \quad \begin{array}{ccc} \cdot & \xrightarrow{Li} & \cdot \\ {\scriptstyle i}\downarrow & \nearrow_{s} & \downarrow{\scriptstyle Ri} \\ \cdot & = & \cdot \end{array} \qquad\qquad f \in \mathcal{R} \quad \text{iff} \quad \begin{array}{ccc} \cdot & = & \cdot \\ {\scriptstyle Lf}\downarrow & \nearrow^{t} & \downarrow{\scriptstyle f} \\ \cdot & \xrightarrow[Rf]{} & \cdot \end{array}$$

Constructing lifts

Given a coalgebra (i, s) and an algebra (f, t), any lifting problem

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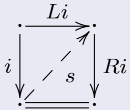
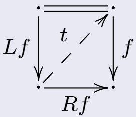
$i \in \mathcal{L}$ iff 

$f \in \mathcal{R}$ iff 

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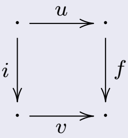

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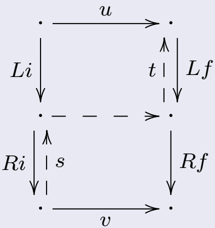

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$\mathbb{L}$-coalgebras lift against $\mathbb{R}$-algebras—but so do (L, ϵ)-coalgebras and (R, η)-algebras. Hence the underlying wfs has

$$\mathcal{L} = \text{ retract closure of the } \mathbb{L}\text{-coalgebras}$$
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Lemma (R.)

In a cofibrantly generated awfs, all right maps admit $\mathbb{R}$-algebra structures.

Cofibrantly generated algebraic weak factorization systems

Cofibrantly generated wfs

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- morphisms of awfs $(\mathbb{L}, \mathbb{R}) \to (\mathbb{L}', \mathbb{R}') \leftrightsquigarrow \mathcal{J} \to \mathbb{L}'\text{-}\mathbf{coalg}$
- i.e., a morphism exists iff the generators $\mathcal{J}$ are cellular for $\mathbb{L}'$.

A sample theorem

Theorem (R.)

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$$\begin{array}{ccccc} |\partial\Delta^n| & \cong & S^{n-1} & \longrightarrow & X \\ \downarrow & & \downarrow & \nearrow & \downarrow f \\ |\Delta^n| & \cong & D^n & \longrightarrow & Y \end{array}$$

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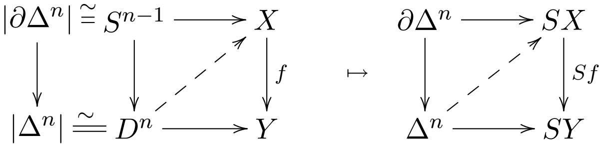

- map via S to algebraic trivial fibrations with chosen sphere fillers

Toward adjunctions of awfs

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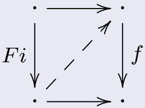

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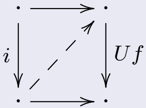

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One way to make this precise uses the theory of mates. Alternatively ...

Awfs encoded as double categories

Lemma (Garner)

An awfs $(\mathbb{L}, \mathbb{R})$ gives rise to and can be recovered from either of two double categories $\mathbb{C}\mathbf{oalg}(\mathbb{L})$ or $\mathbb{A}\mathbf{lg}(\mathbb{R})$.

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There is a forgetful double functor $\mathbb{A}\mathbf{lg}(\mathbb{R}) \to \mathbb{S}\mathbf{q}(\mathcal{M})$.

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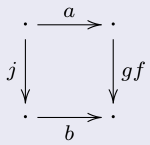
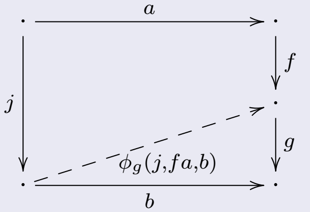
Define ϕ_{gf} by solving 

via 

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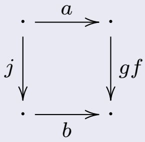
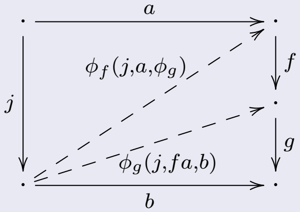
Define ϕ_{gf} by solving  via 

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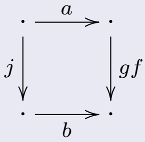

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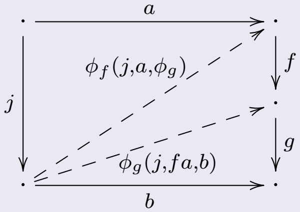

This composition law encodes the comultiplication for $\mathbb{L}$ (and dually).

Adjunctions of algebraic weak factorization systems

Lemma/Definition

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Corollary (composition criterion)

A lifted right adjoint $U\colon \mathbb{R}'\textbf{-alg} \to \mathbb{R}\textbf{-alg}$ defines an adjunction of awfs iff it preserves vertical composition of algebras.

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The functor $\mathcal{J} \to \mathbb{L}\textbf{-coalg}$ constructed by Garner's small object argument is universal among adjunctions of awfs.

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- each functor preserves vertical composition

Two-variable adjunctions and enrichment

Definition

A two-variable adjunction consists of pointwise adjoint bifunctors

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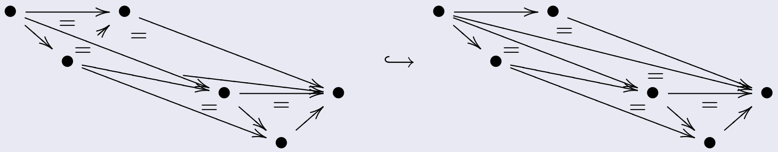
Induced two-variable adjunctions

$(\hat{\otimes}, \hat{\hom}_\ell, \hat{\hom}_r)\colon \mathcal{K}^{\mathbf{2}} \times \mathcal{M}^{\mathbf{2}} \to \mathcal{N}^{\mathbf{2}}$ e.g., $(\Lambda^2_1 \to \Delta^2)\hat{\otimes}(\partial\Delta^1 \to \Delta^1)$ is



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$$\begin{aligned} -\hat{\otimes}- \colon \mathbb{L}\textbf{-coalg} \times \mathbb{L}'\textbf{-coalg} &\to \mathbb{L}''\textbf{-coalg} \\ \hat{\hom}_\ell(-,-) \colon \mathbb{L}\textbf{-coalg}^{\mathrm{op}} \times \mathbb{R}''\textbf{-alg} &\to \mathbb{R}'\textbf{-alg} \\ \hat{\hom}_r(-,-) \colon \mathbb{L}'\textbf{-coalg}^{\mathrm{op}} \times \mathbb{R}''\textbf{-alg} &\to \mathbb{R}\textbf{-alg} \end{aligned}$$

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Sadly, the lifted functors don't even preserve *composability* of (co)algebras.

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A lifted functor $\hat{\text{hom}}(-,-)\colon \mathbb{L}'\textbf{-coalg}^{\text{op}} \times \mathbb{R}''\textbf{-alg} \to \mathbb{R}\textbf{-alg}$ determines a two-variable adjunction of awfs iff,

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$$\begin{array}{ccc} K & \xrightarrow{\quad a \quad} & X^B \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \hat{\hom}(i,gf)} \\ L & \xrightarrow[\quad b \times c \quad]{} & Z^B \times_{Z^A} X^A \end{array}$$

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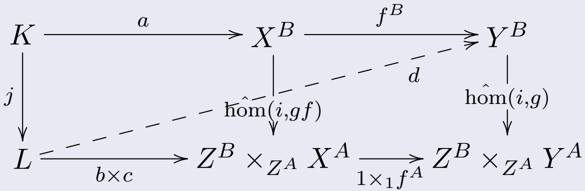

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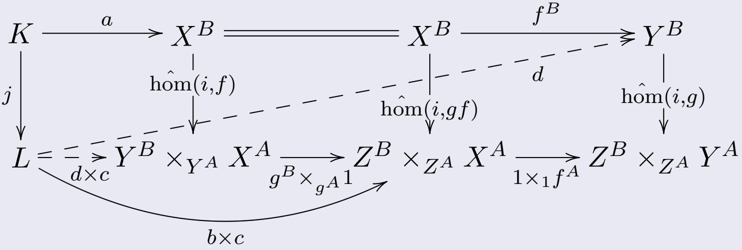

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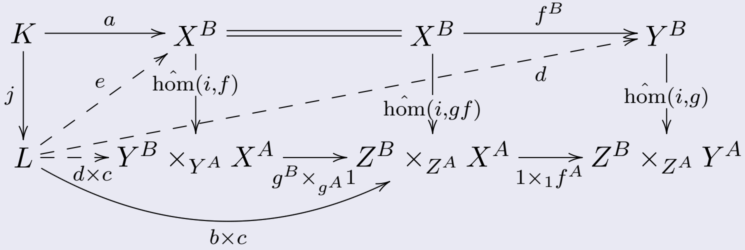

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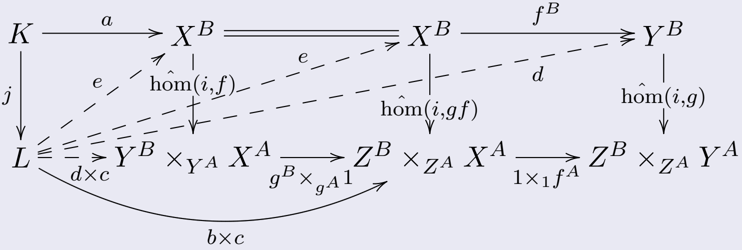

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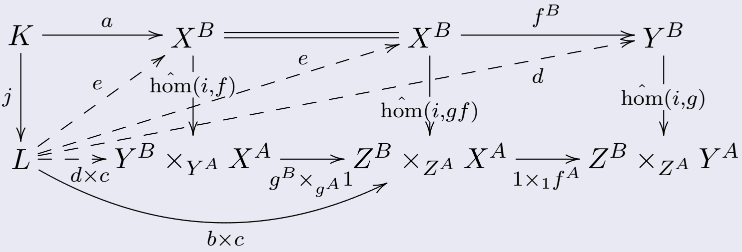

and also satisfies a dual condition in the first variable.

The cellularity theorem

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Given a two-variable adjunction $\otimes\colon \mathcal{K} \times \mathcal{M} \to \mathcal{N}$, awfs $(\mathbb{L}, \mathbb{R})$ and $(\mathbb{L}', \mathbb{R}')$ on $\mathcal{K}$ and $\mathcal{M}$ generated by $\mathcal{J}$ and $\mathcal{J}'$, and an awfs $(\mathbb{L}'', \mathbb{R}'')$ on $\mathcal{N}$,

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- $\otimes$ is a two-variable adjunction of awfs iff $\mathcal{J} \hat{\otimes} \mathcal{J}'$ is cellular, i.e., iff there exists

$$\begin{array}{ccc} \mathcal{J} \times \mathcal{J}' & \dashrightarrow & \mathbb{L}''\textbf{-coalg} \\ \downarrow & & \downarrow \\ \mathcal{K}^{\mathbf{2}} \times \mathcal{M}^{\mathbf{2}} & \xrightarrow{\hat{\otimes}} & \mathcal{N}^{\mathbf{2}} \end{array}$$

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- Furthermore, the two-variable adjunction of awfs is determined by the coalgebra structures assigned to elements of $\mathcal{J}\hat{\otimes}\mathcal{J}'$.

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Sample Theorems (R.)

Quillen's model structure on $\mathbf{sSet}$ and the folk model structure on $\mathbf{Cat}$ are (cartesian) monoidal algebraic model structures.

Acknowledgments

Thanks

Thanks to the organizers, Eugenia Cheng, Richard Garner, Martin Hyland, Peter May, Mike Shulman, and the members of the category theory seminars at Chicago, Macquarie, and Sheffield.

Further details

Further details can be found in

- "Algebraic model structures" New York J. Math **17** (2011) 173-231
- "Monoidal algebraic model structures" a preprint available at `www.math.uchicago.edu/~eriehl`
- my Ph.D. thesis "Algebraic model structures" available at `www.math.uchicago.edu/~eriehl`