Twisted Equivariant Cohomology and Orbifolds

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Outline

Equivariant Homotopy

Fix a (compact Lie) group G.

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Th A G-map $f : X \to Y$ is a G-homotopy equivalence iff $f^H : X^H \to Y^H$ is a homotopy equivalence for all (closed) subgroups H

We think of G-spaces as diagrams of fixed sets.

Orbit Category

These diagrams are organized by

$\mathcal{O}_G$: category with

objects the canonical orbits G/H
morphisms are G-maps

A map $G/H \to X$ is equivalent to $x \in X^H$ such that $eH \to x$

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Maps in $\mathcal{O}_G$ are defined by multiplication by elements $\alpha \in G$

Orbit Category

Eg $G = \mathbb{Z}/2$

$\mathcal{O}_G$ has two types of orbits (free or fixed) so there are two objects, G/G and G/e

two non-identity maps:

- a non-trivial self-map $G/e \to G/e$
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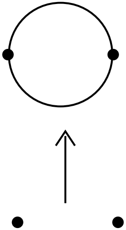
$$\tau^2 = id$$

$$\tau\rho = \rho$$

Example

Eg $X = S^1$ with $G = \mathbb{Z}/2$ action



X^G is two points.

Algebraic Invariants

We can apply an algebraic invariant to the diagram of fixed sets

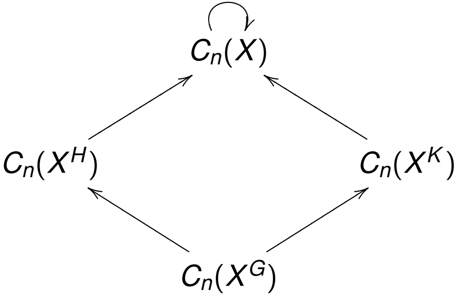
The result will be a diagram of Abelian groups (or whatever)

ie. (contravariant) functors $\mathcal{O}_G \to Ab$

Diagrams of Fixed Sets

Bredon cohomology:

Apply the chain complex functor C_* to the diagram $\{X^H\}$



If $\underline{A}$ is a *coefficient system*, a functor $\underline{A} : \mathcal{O}_G^{op} \to Ab$, then $Hom(\underline{C}_*(X), \underline{A})$ is a chain complex

We define $H_G^*(X) = H_{\mathcal{O}_G}^*(Hom(\underline{C}_*(X), \underline{A}))$

Fundamental Groupoid

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Wrap up into a single category:

Grothendieck semidirect product construction

Start with a functor $F : \mathcal{C}^{op} \to \mathbf{Cat}$

To combine image categories $F(C)$ into one category:

The Grothendieck semidirect product $\int_{\mathcal{C}} F$

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$\int_{\mathcal{C}} F$ has projection functor to $\mathcal{C}$

turns a functor $F : \mathcal{C}^{op} \to \mathbf{Cat}$ into an object in the slice category $\mathbf{Cat}\ /\mathcal{C}$

projection retains information about different $F(C)$

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$\Pi_G(X) = \int_{\mathcal{O}_G} \underline{\Pi}$

- **objects** are pairs $(G/H, x)$ for $x \in \Pi(X^H)$
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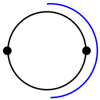
  Defined by tom Dieck

Example: X = S^1 again

$\Pi_G(X)$ contains 'loops' (α, γ) at x from a path $\gamma : x \to \alpha x$



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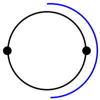

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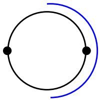

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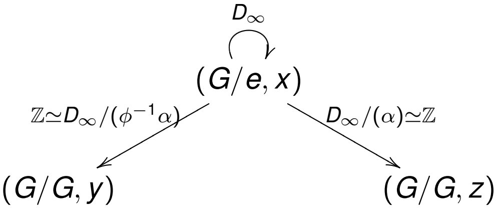
$\Pi_G(X)$ looks something like:



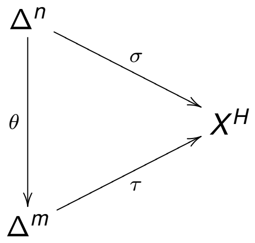
Simplicial Category

(Mordijk-Svensson) $\underline{\Delta}_X(G/H)$ is the category with

objects

$$\left(\sigma \colon \Delta^n \to X^H\right).$$

arrows $\theta \colon (\sigma \colon \Delta^n \to X^H) \to (\tau \colon \Delta^m \to X^H)$ are simplicial operators $\Delta^n \to \Delta^m$ such that the diagram



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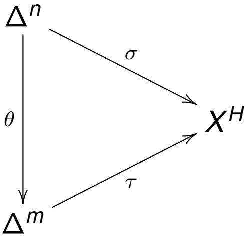

The equivariant simplicial category $\Delta_G(X) = \int_{\mathcal{O}_G} \underline{\Delta}$ calculates Bredon cohomology.

Non-Discrete Groups

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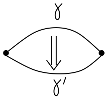
Goal: arrange definitions to discard topology of G, but retain topology of action of G on X

In constructing algebraic invariants, we want to index over $h\mathcal{O}_G$

In Bredon cohomology, take chains of $X^H/(WH)_0$

Disrete Equivariant Fundamental Groupoid

Instead of taking homotopy classes of paths, we take $\Pi(X)$ as a 2-category



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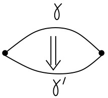
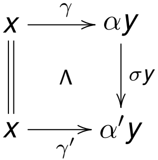


2-category version of $\int_{\mathcal{O}_G} \underline{\Pi}$:

A 2-cell is a pair (σ, Λ)

$\sigma : \alpha \Rightarrow \alpha'$ in $\mathcal{O}_G$ and $\Lambda : \gamma \Rightarrow \gamma' \circ \sigma$



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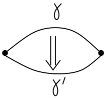

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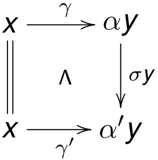


Identifying 2-cells gives $\Pi_G^d(X)$ (also tom Dieck)
Remove relabelling loops coming from topology on G

Discrete Simplicial Category

If $H \leq G$ a homotopy $\Lambda\colon D \times I \to X^H$ *comes from G* if there exists a homotopy

$$\Gamma\colon D \times I \to G/H$$

such that

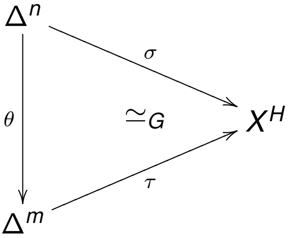
$$\begin{aligned} \Gamma(x,0) &= eH && \text{for all } x \in D, \\ \Lambda(x,s) &= \Gamma(x,s)\Lambda(x,0) && \text{for all } x \in D \text{ and } 0 \leq s \leq 1. \end{aligned}$$

Let $\underline{\Delta}_X(G/H)$ be the category with

objects

$$\left(\sigma\colon \Delta^n \to X^H\right)$$

arrows $\theta\colon (\sigma\colon \Delta^n \to X^H) \to (\tau\colon \Delta^m \to X^H)$ are simplicial operators $\Delta^n \to \Delta^m$ such that the diagram



commutes up to a homotopy coming from G.

Define a 2-functor $\underline{\Delta}_X \colon \mathcal{O}_G \to$ **Cat**

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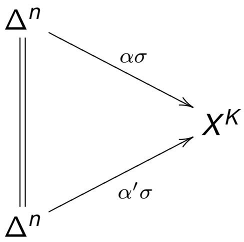
2-cells: a homotopy γ from α to α' in $\mathcal{O}_G$ gives a natural transformation with components $\Delta_X(\gamma)_{(\sigma,\Delta^n,K)} = (\mathrm{id}_{\Delta^n})$

This works because the diagram

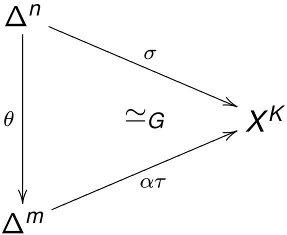


commutes up to the homotopy $\gamma\sigma$, which comes from G.

Form the Grothendieck 2-category $\Delta_G(X) = \int_{\mathcal{O}_G} \underline{\Delta}_X$ over $\mathcal{O}_G$:

objects $(G/H, \sigma\colon \Delta^n \to X^H)$

arrows (α, θ) where $\alpha\colon G/H \to G/K$ is an arrow in $\mathcal{O}_G$, and $\theta\colon \Delta^n \to \Delta^m$ a simplicial operator, such that the diagram



commutes up to homotopy from G.

2-cells of $\Delta_G(X)$ are of the form

$$(\gamma)\colon (\alpha, \theta) \Rightarrow (\alpha', \theta)$$

where γ is a path from α to α' in $\mathcal{O}_G$.

Form the discrete simplicial category $\Delta^d_G(X)$:

mod out by the 2-cells of $\Delta_G(X)$):

$([\alpha], \theta) = ([\alpha'], \theta')$ when $[\alpha] = [\alpha']$ in $h\mathcal{O}_G$ and $\theta = \theta'$.

This lies over $h\mathcal{O}_G$.

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There are functors $\Delta_G(X) \to \Pi_G(X)$ and $\Delta_G^d(X) \to \Pi_G^d(X)$ which lie over $\mathcal{O}_G$ and $h\mathcal{O}_G$ respectively in the sense that the following triangles commute:

$$\begin{array}{ccc} \Delta_G(X) & \longrightarrow & \Pi_G(X) \\ & \searrow \quad \swarrow & \\ & \mathcal{O}_G & \end{array} \qquad \begin{array}{ccc} \Delta_G^d(X) & \longrightarrow & \Pi_G^d(X) \\ & \searrow \quad \swarrow & \\ & h\mathcal{O}_G & \end{array}$$

The cohomology of the category $\Delta^d_G(X)$ is the equivariant cohomology of X as a G-space.

Coefficients for this cohomology are contravariant functors $\underline{A}\colon \Delta^d_G(X)^{\mathrm{op}} \to \mathbf{Ab}$.

coefficients are called

constant (with respect to X) when it factors through $h\mathcal{O}_G$ and

local if it factors through $\Pi^d_G(X)$.

Th (Pronk-S) This definition agrees with the usual definition of Bredon cohomology.

Connection to Orbifolds

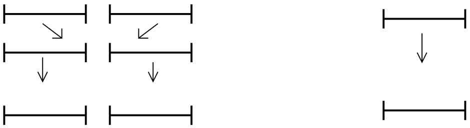
Many orbifolds can be represented by a compact Lie group acting on a manifold with finite isotropy $G \ltimes M$

This representation is not unique:

Change-of-group Morita equivalences

A If $K \subseteq G$ acts freely on X, then
$G \ltimes X$ is equivalent to $G/K \ltimes X/K$

$\mathbb{Z}/2 \ltimes (I \coprod I)$ $\{e\} \ltimes I$



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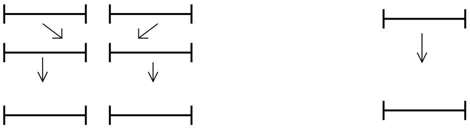

B If $H \subseteq G$, then
$H \ltimes Y$ is equivalent to $G \ltimes (G \times_H Y)$

Theorem from Previous Paper

Define a (bi)category of fractions $\textbf{TrOrbiGrpd}[W^{-1}]$:

- **objects** are transformation groupoids
- **morphisms** are spans $G \ltimes X \xleftarrow{w} K \ltimes Y \xrightarrow{\phi} H \ltimes Z$ where
  - w is a change-of-groups (**A** or **B**)
  - ϕ is an equivariant map

i.e. to define a map, replace $G \ltimes X$ by an equivalent $K \ltimes Y$ and then map out.
Note that this means all change-of-groups are isos.

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Th [Pronk–S] There is an equivalence of bicategories

$$\mathbf{rOrbiGrpd}[W^{-1}] \simeq \mathbf{TrOrbiGrpd}[W^{-1}]$$

representable orbifolds $\longleftrightarrow$ (orbi) G-spaces
up to change of groups

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So we have orbifold invariants and a way of defining Bredon cohomology with twisted coefficeints for orbifolds.