Exact completion and small sheaves

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Outline

1. Background and motivation
2. Exact completion of unary sites
3. Pretopos completions of higher-ary sites
4. Small sheaves

Exact completions

Recall: a (Barr-)exact category has finite limits and quotients of equivalence relations, which interact as they do in $\mathcal{S}et$.

The following forgetful functors have left adjoints:

- Exact categories $\hookrightarrow$ regular categories
- Exact categories $\to$ lex categories (= finitely-complete)
- Regular categories $\to$ lex categories

References: Carboni & Celia Magno, Carboni & Vitale, Hu, Hu & Tholen, Lawvere, Succi Cruciani, Freyd & Scedrov, Lack, Karazeris …

The ex/reg completion

Let **C** be regular. Then $\mathbf{C}_{\mathrm{ex/reg}}$ has

- objects: equivalence relations in **C**.
- morphisms: relations in **C** which are equivalence-respecting, entire, and functional.

Constructed by splitting symmetric monads in the allegory of relations.

Or: if **C** is small, $\mathbf{C}_{\mathrm{ex/reg}}$ is the full subcategory of $\mathrm{Sh}(\mathbf{C}, J_{\mathrm{reg}})$ spanned by the quotients of equivalence relations in **C**.

The ex/lex completion

Let **C** be lex. Then $\mathbf{C}_{\mathrm{ex/lex}}$ has

- objects: “pseudo-equivalence relations” in **C**.
- morphisms: equivalence classes of equivalence-respecting morphisms in **C**.

Constructed by splitting symmetric monads in the allegory of “relations” in the preorder reflections of slice categories.

Or: if **C** is small, $\mathbf{C}_{\mathrm{ex/lex}}$ is the full subcategory of $\mathrm{Psh}(\mathbf{C})$ spanned by the quotients of pseudo-equivalence relations in **C**.

Question 1: The ex/wlex completion?

The ex/lex construction works just as well when **C** only has *weak* finite limits (which satisfy the existence, but not the uniqueness, part of the usual universal property).

But the result is *not* left adjoint to the forgetful functor

$$\text{exact categories} \to \text{weakly lex categories!}$$

Instead it classifies "left covering functors" (Carboni & Vitale).

Question 2: Other topologies?

What is special about

1. the regular topology on a regular category, and
2. the trivial topology on a weakly lex category,

so that we can find “exact completions” inside their categories of sheaves?

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Exact completion of sites

Theorem

There is a 2-category of unary sites, in which exact categories form a full reflective sub-2-category.

The reflector:

- on a regular category with its regular topology, constructs its ex/reg completion;
- on a weakly lex category with its trivial topology, constructs its ex/wlex completion.

What is a unary site?

Definition

A site is unary if

1. Its topology is generated by singleton covers (every covering sieve contains a covering sieve that is generated by a single morphism), and
2. It has local weak finite limits.

A local weak limit of a diagram G in a site is

1. A cone $T\colon x \Rightarrow G$ such that
2. For every other cone $S\colon z \Rightarrow G$, there exists a covering family $\{p_i\colon w_i \to z\}$ such that each cone $S \circ p_i$ factors through T.

Examples

- The regular topology on a regular category;
- The trivial topology on a weakly lex category.

What is a morphism of unary sites?

Theorem

Let $\mathbf{C}, \mathbf{D}$ be unary sites. For a functor $F \colon \mathbf{C} \to \mathbf{D}$, the following are equivalent.

1. *F preserves local weak finite limits.*
2. *F preserves covers, and is flat relative to the topology of $\mathbf{D}$.*
3. *(If $\mathbf{C}$ is lex and $\mathbf{D}$ is subcanonical) F is lex and preserves covers.*

These are the morphisms of (unary) sites.

Definition (Karazeris)

$F \colon \mathbf{C} \to \mathbf{D}$ is *flat relative to the topology of* $\mathbf{D}$ if for any finite diagram G in $\mathbf{C}$, and any cone $S \colon z \Rightarrow FG$ in $\mathbf{D}$, there is a covering family $\{p_i \colon w_i \to z\}$ such that each cone $S \circ p_i$ factors through $F(T)$ for some cone $T \colon x \Rightarrow G$ in $\mathbf{C}$.

What is a morphism of unary sites?

Examples

- Between regular categories: regular functors.
- Between (weakly) lex categories: (weakly) lex functors.
- From weakly lex categories to exact categories: left covering functors (Karazeris).

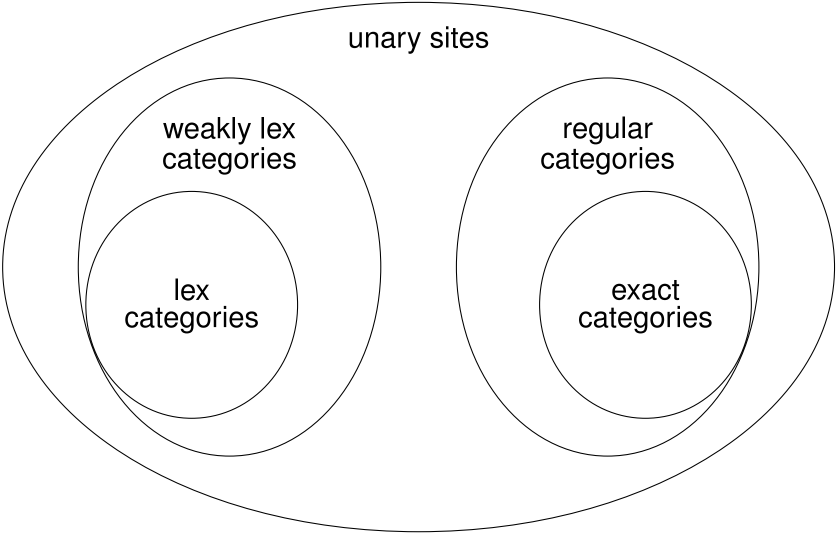
The universal property

For a unary site **C**, and an exact category **D**,

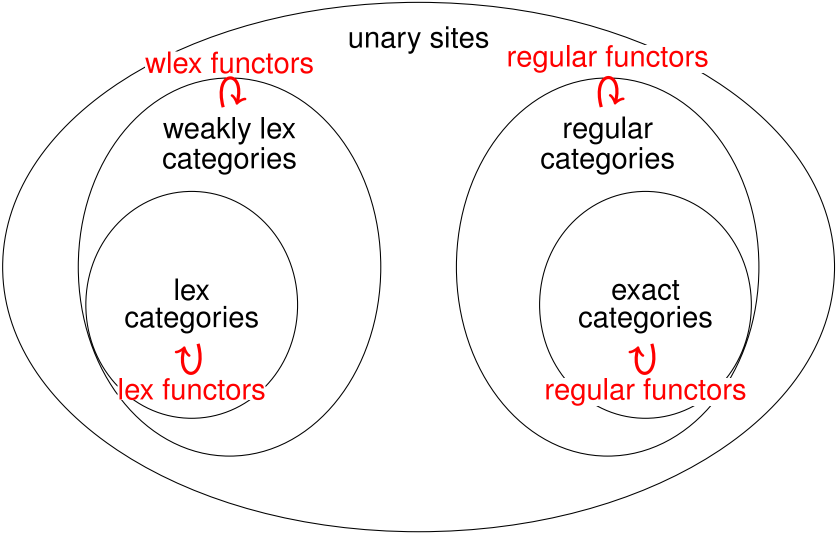
$$\frac{\text{morphisms of sites } \mathbf{C} \to \mathbf{D}}{\text{regular functors } \mathbf{C}_{\mathrm{ex}} \to \mathbf{D}}$$

where $\mathbf{C}_{\mathrm{ex}}$ and **D** have their regular topologies.

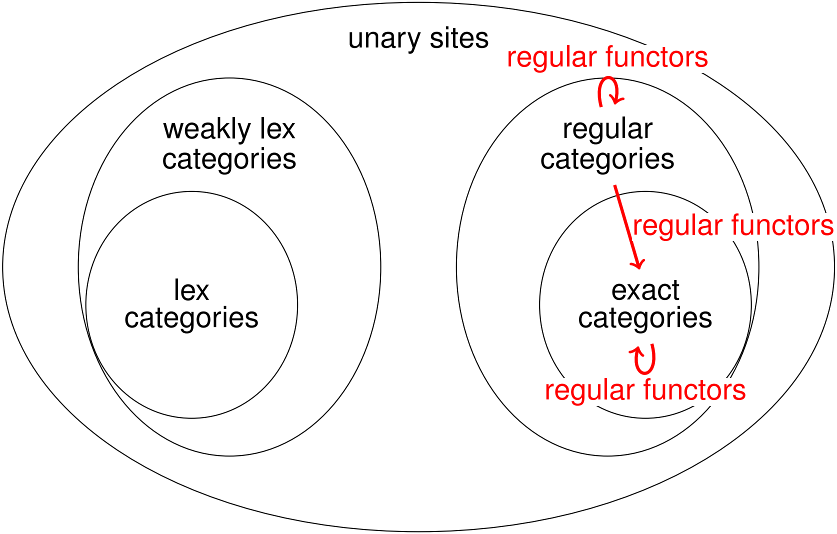
The 2-category of unary sites



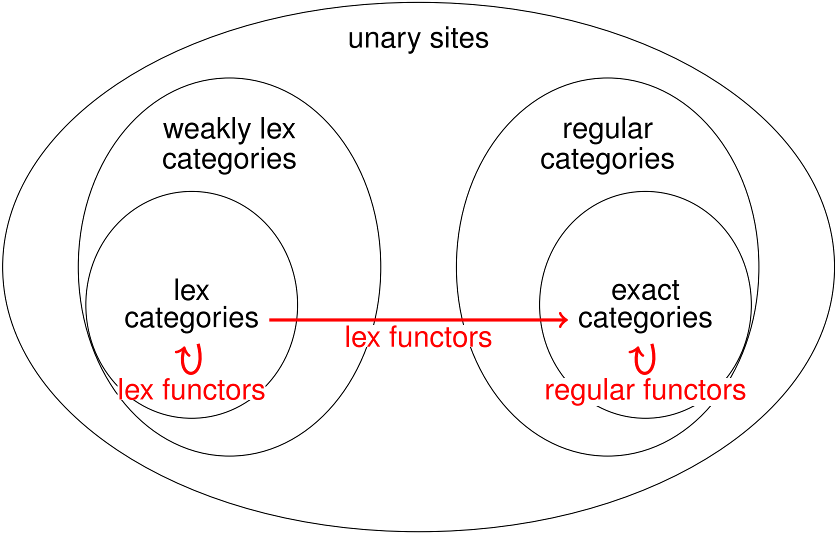
The 2-category of unary sites



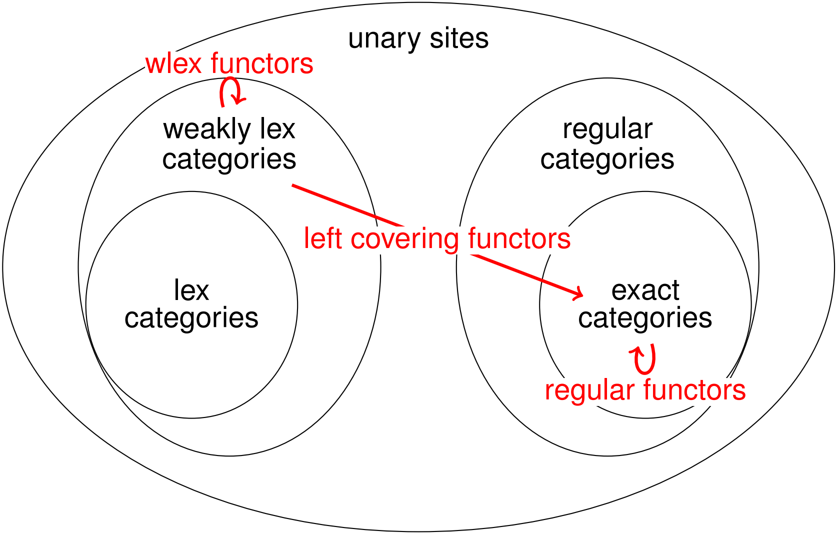
The 2-category of unary sites



The 2-category of unary sites



The 2-category of unary sites



Constructing the exact completion

Let **C** be a unary site. Then $\mathbf{C}_{\mathrm{ex}}$ has

- objects: "equivalence relations" in **C**, modulo its topology.
- morphisms: either
  - relations in **C** which are equivalence-respecting, entire, and functional (modulo the topology); or
  - a category of fractions of equivalence-respecting morphisms in **C**.

Can be constructed by splitting symmetric monads in a suitable allegory of relations.

Or: if **C** is small, $\mathbf{C}_{\mathrm{ex}}$ is the full subcategory of $\mathrm{Sh}(\mathbf{C})$ spanned by the quotients of such equivalence relations in **C**.

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κ-ary pretoposes

Let κ be a regular cardinal, or the size of the universe "∞".

Definition

A κ-ary pretopos is an exact category which is also κ-ary extensive (has disjoint and stable coproducts of size $< \kappa$).

Examples

- A ω-ary pretopos is usually called just a "pretopos".
- An ∞-ary pretopos (or "∞-pretopos" or "faux topos") is a category which satisfies all the exactness conditions of Giraud's theorem.
- A 2-ary pretopos is an exact category with a strict initial object.

κ-ary sites

Definition

A site is κ-ary if

- its topology is generated by families of size $< \kappa$, and
- it has "local weak finite κ-multilimits". That is, every finite diagram has a κ-small family of cones through which every other cone factors modulo passage to a covering family.

Examples

- The κ-canonical topology on a κ-ary pretopos is κ-ary.
- Every κ-ary site is λ-ary for any $\lambda \geq \kappa$.
- Every *small* site is ∞-ary.

Morphisms of κ-ary sites

Theorem

Let **C**, **D** *be κ-ary sites. For a functor F:* **C** $\to$ **D***, the following are equivalent.*

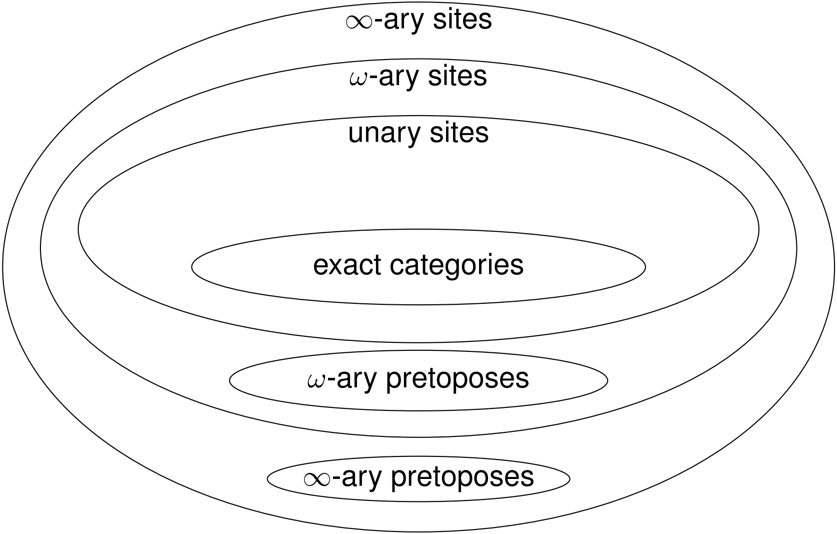
1. *F preserves local weak finite κ-multilimits*
2. *F preserves covers, and is flat relative to the topology of* **D**
3. *(If* **C** *is lex and* **D** *is subcanonical) F is lex and preserves covers.*

These are the morphisms of (κ-ary) sites.

Remarks

- Independent of κ.
- Between Grothendieck topoi: inverse image functors.
- From a small site **C** to a topos **D**: the functors which $\mathrm{Sh}($**C**$)$ classifies.

The 2-categories of κ-ary sites



κ-ary pretopos completion

Theorem

The 2-category of κ-ary sites contains the 2-category of κ-ary pretoposes as a full reflective sub-2-category.

The reflector:

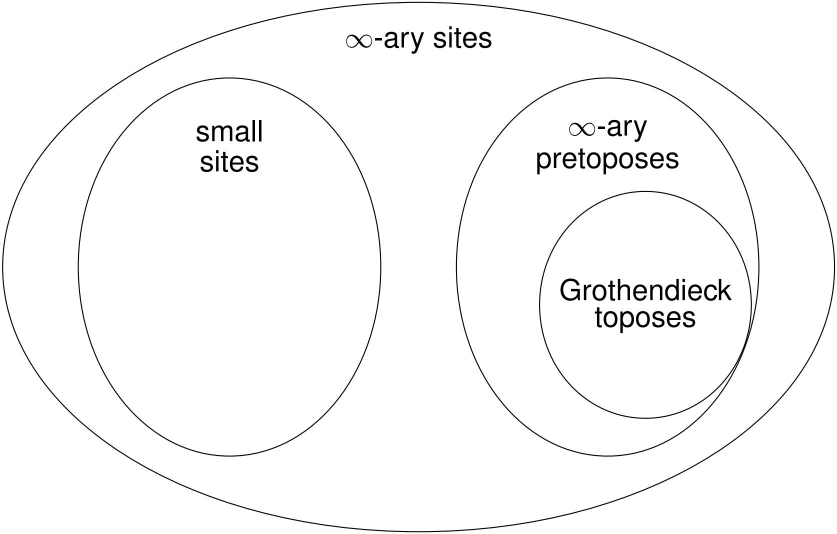
- on lex and coherent categories, constructs pretop/lex and pretop/coh completions;
- on a small (∞-ary) site, constructs its topos of sheaves.

The universal property:

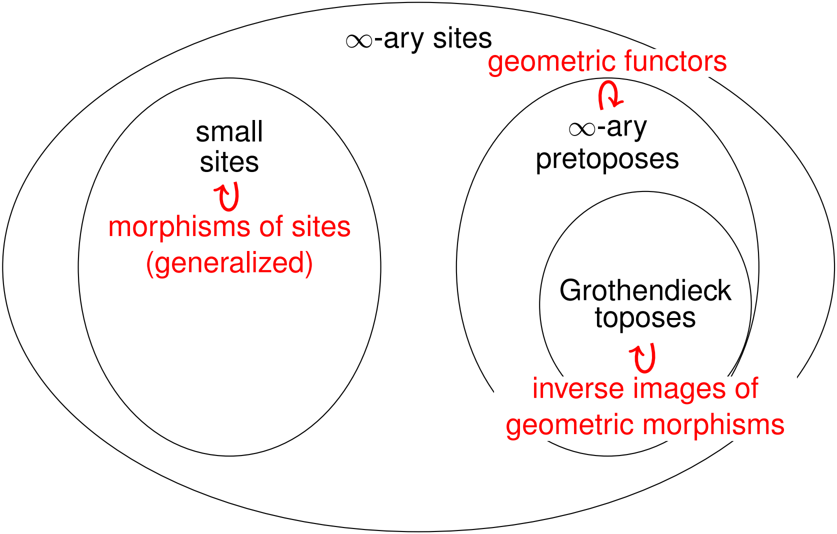
$$\frac{\text{morphisms of sites } \mathbf{C} \to \mathbf{D}}{\kappa\text{-coherent functors } \mathbf{C}_{\text{ex}} \to \mathbf{D}}$$

where $\mathbf{C}_{\text{ex}}$ and $\mathbf{D}$ have their κ-canonical topologies.

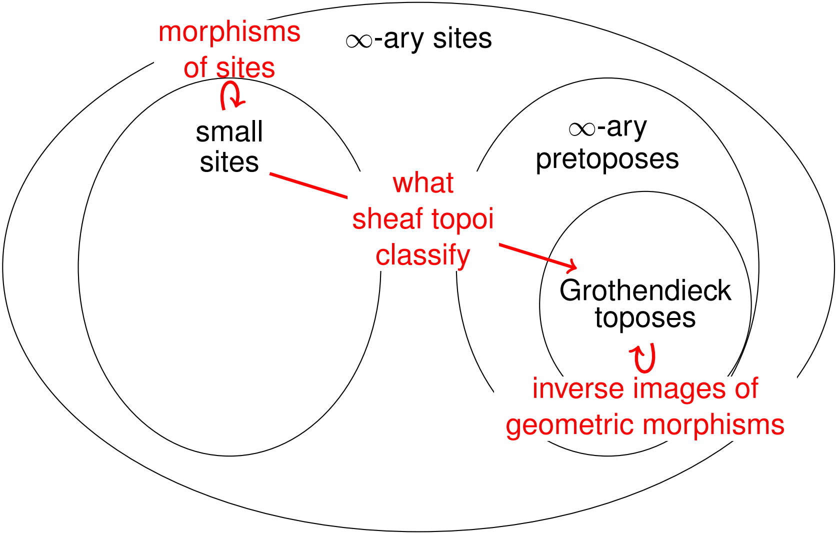
The 2-category of ∞-ary sites



The 2-category of ∞-ary sites



The 2-category of ∞-ary sites



Constructing the pretopos completion

Let **C** be a κ-ary site. Then $\mathbf{C}_{\kappa\text{-pretop}}$ has

- objects: "$(<\kappa)$-object equivalence relations" in **C**, modulo its topology.
- morphisms: either
    - $(<\kappa)$-object equivalence-respecting relations which are entire and functional (modulo the topology); or
    - a category of fractions of equivalence-respecting families of morphisms in **C**.

Can be constructed by adjoining κ-ary coproducts and splitting symmetric monads in a suitable allegory of relations.

Or: if **C** is small, $\mathbf{C}_{\kappa\text{-pretop}}$ is the full subcategory of $\mathrm{Sh}(\mathbf{C})$ spanned by the quotients of such many-object equivalence relations in **C**.

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Small presheaves

Let **C** be a large category.

Definition

A small presheaf on **C** is a presheaf $\mathbf{C}^{op} \to \mathrm{Set}$ which is a small colimit of representables.

The category $\mathcal{P}\mathbf{C}$ of small presheaves on **C** is its free cocompletion under small colimits.

Now suppose **C** has weak finite ∞-multilimits, so that its trivial topology is ∞-ary. Day and Lack proved this is equivalent to $\mathcal{P}\mathbf{C}$ being lex. But in fact:

Theorem

*In this case, $\mathcal{P}\mathbf{C}$ is equivalent to the ∞-ary pretopos completion of **C**. In particular, it is an ∞-ary pretopos.*

Small sheaves

Let **C** be a large ∞-ary site.

Definition

A small sheaf on **C** is an object of its ∞-ary pretopos completion.

Example

$\mathbf{C} = \mathrm{Ring}^{\mathrm{op}}$ with the Zariski topology. Then a small sheaf is a many-object equivalence relation in **C**: a family of rings with information about how to glue them together. Any scheme can be seen as such an object.

Thanks!