Classifying Fiber Bundles with Fiber $K(\pi, n)$

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Postnikov Systems

A *Postnikov system* for $X \in \mathbf{S}$ is a tower of spaces

$$\ldots \to X^n \xrightarrow{q^n} X^{n-1} \to \ldots X^1 \to X^0$$

together with maps $p^n : X \to X^n$ such that $p^{n-1} = q^n p^n$, p^n is an n-equivalence, and X^n is an n-type.

X Kan makes X^n Kan and p^n & q^n Kan fibrations.

K^n is the fiber of q^n. $\pi_i(K^n) = 0 \; i \neq n \; \pi_n(K^n) = \pi_n(X)$. K^n is an *Eilenberg-Mac Lane space* $K(\pi_n(X), n)$.

$X \to \varprojlim X^n$ is a weak equivalence, so the tower (X^n) is a "construction" of X one homotopy group at a time. $\pi_n(X)$ is added by q^n. How? What are the q^n ?

k-Invariants

If X is minimal (technical, no loss of generality) the q^n are locally trivial fiber bundles with fiber $K(\pi_n(X), n)$.
If X is simply connected ($\pi_1(X) = 1$, lots of loss of generality) the q^n are *principal* $K(\pi_n(X), n)$ bundles, i.e. torsors over X^{n-1} for the simplicial abelian group $K(\pi_n(X), n)$.
In that case, we have the $K(\pi_n(X), n)$ torsor (more later)

$$L(\pi_n(X), n+1) \to K(\pi_n(X), n+1)$$

with $L(\pi_n(X), n+1)$ contractible, which makes it *universal*. So there are maps $k^{n+1} : X^{n-1} \to K(\pi_n(X), n+1)$ such that

$$\begin{array}{ccc} X^n & \longrightarrow & L(\pi_n(X), n+1) \\ {\scriptstyle q^n}\downarrow & & \downarrow \\ X^{n-1} & \xrightarrow[k^{n+1}]{} & K(\pi_n(X), n+1) \end{array}$$

is a pullback. The k^{n+1} are the *k-invariants* of X. They determine X up to homotopy.

π_1-Actions

When $\pi_1 \neq 1$ there are π_1-actions that must be taken into account.

For example, let $p : E \to B$ be a fibration with B connected. Then the fundamental groupoid $\pi_1(B)$ acts on the homotopy of the fibers of p:

$$\mathbf{S}/B \approx Set^{(\Delta/B)^{op}} \underset{q*}{\overset{\pi_0^B}{\rightleftarrows}} Set^{G(\Delta/B)^{op}}$$

$G(\Delta/B) \approx \pi_1(B)$. $\pi_0^B \dashv q^*$ is fiberwise π_0.

Suppose the fibers of p are simply connected - so we don't have to worry about basepoints when defining homotopy groups.

$$\pi_n^B(p) = \pi_0^B(p^{S^n})$$

Write $\tilde{\pi}_n$ for this. $\tilde{\pi}_n$ is an abelian group in $Set^{G(\Delta/B)^{op}}$ for $n \geq 2$.

Eilenberg-Mac Lane Spaces

$$N : s(Ab) \longleftrightarrow Ch_+(Ab) : D$$

is the Dold-Kan correspondence

$$Dk(\pi, n) = K(\pi, n) \text{ and } Dl(\pi, n+1) = L\pi, n+1)$$

Dold-Puppe: *Ab* can be replaced above by *any* abelian category. So let $p : E \to B$ be a locally trivial bundle with fiber $K(\pi, n)$ and B connected. Let $\pi_1 = \pi_1(B)$ and denote by $\tilde{\pi} \in Set^{\pi_1^{op}}$ the n-dimensional fiberwise homotopy of p.

$$N : s(Ab(Set^{\pi_1^{op}})) \longleftrightarrow Ch_+(Ab(Set^{\pi_1^{op}})) : D$$

is the Dold-Kan correspondence

$$Dk(\tilde{\pi}, n) = K(\tilde{\pi}, n) \text{ and } Dl(\tilde{\pi}, n+1) = L\tilde{\pi}, n+1)$$

All in $\mathbf{S}^{\pi_1^{op}}$.

Moving over $K(\pi_1, 1)$

The functor

$$(\ ) \otimes_{\pi_1} L(\pi_1, 1) : \mathbf{S}^{\pi_1^{op}} \to \mathbf{S}/K(\pi_1, 1)$$

has many nice properties:

- It has left and right adjoints

The left adjoint is pulling back over $L(\pi_1, 1) \to K(\pi_1, 1)$, which applied to the cononical map $B \to K(\pi_1, 1)$ gives $\tilde{B} \to B$ the universal cover of B.

- It preserves and reflects weak equivalences
- It preserves cofibrations and fibrations
- The unit and counit of the left adjoint pair are weak equivalences

So $(\ )\otimes_{\pi_1} L(\pi_1,1)$ is a left and right Quillen functor and induces an equivalence on the homotopy categories.

$K(\tilde{\pi},n)$ is an abelian group in $\mathbf{S}^{\pi_1^{op}}$, so $K(\tilde{\pi},n)\otimes_{\pi_1} L(\pi_1,1)$ is an abelian group in $\mathbf{S}/K(\pi_1,1)$.

$L(\tilde{\pi},n+1)\to K(\tilde{\pi},n+1)$ is a $K(\tilde{\pi},n)$ torsor, so $L(\tilde{\pi},n+1)\otimes_{\pi_1} L(\pi_1,1)\to K(\tilde{\pi},n+1)\otimes_{\pi_1} L(\pi_1,1)$ is an $K(\tilde{\pi},n)\otimes_{\pi_1} L(\pi_1,1)$ torsor in $\mathbf{S}/K(\pi_1,1)$.

$L(\tilde{\pi},n+1)\to 1$ is a weak equivalence and $K(\tilde{\pi},n+1)$ is fibrant, so $L(\tilde{\pi},n+1)\otimes_{\pi_1} L(\pi_1,1)\to K(\pi_1,1)$ is a weak equivalence and $K(\tilde{\pi},n+1)\otimes_{\pi_1} L(\pi_1,1)\to K(\pi_1,1)$ is a fibration.

It follows that
$L(\tilde{\pi},n+1)\otimes_{\pi_1} L(\pi_1,1)\to K(\tilde{\pi},n+1)\otimes_{\pi_1} L(\pi_1,1)$ is the universal $K(\tilde{\pi},n)\otimes_{\pi_1} L(\pi_1,1)$ torsor in $\mathbf{S}/K(\pi_1,1)$.

The Main Result

Let $p : E \to B$ be a bundle with fiber $K(\pi, n)$ $n \geq 2$, and B connected. Write $K(\tilde{\pi}, n)$ for $K(\tilde{\pi}, n) \otimes_{\pi_1} L(\pi_1, 1)$ and also for its pullback over B.

Theorem

If p has a section $s : B \to E$ with $ps = id$, then there is a unique isomorphism

$$\begin{array}{ccc} E & \xrightarrow{\phi} & K(\tilde{\pi}, n) \\ & {\scriptstyle p}\searrow \quad \swarrow & \\ & B & \end{array}$$

such that $\phi s = 0$, and ϕ induces the identity on homotopy.

Now $E \times_B E \to E$ has a section $\delta : E \to E \times_B E$ so there is a unique isomorphism $\phi : E \times_B E \to K(\tilde{\pi}, n) \times_B E$.
$\phi^{-1} = (a, \pi_2) : K(\tilde{\pi}, n) \times_B E \to E \times_B E$, where
$a : K(\tilde{\pi}, n) \times_B E \to E$ is an action making p a $K(\tilde{\pi}, n)$ torsor over B. From the above, it follows that there is a pullback

$$\begin{array}{ccc} E & \longrightarrow & L(\tilde{\pi}, n+1) \otimes_{\pi_1} L(\pi_1, 1) \\ {\scriptstyle p}\downarrow & & \downarrow \\ B & \xrightarrow[k]{} & K(\tilde{\pi}, n+1) \otimes_{\pi_1} L(\pi_1, 1) \end{array}$$

all over $K(\pi_1, 1)$.

When p is $q^n : X^n \to X^{n-1}$ from before, this k is the desired k-invariant.

Final Remarks

The full classification theorem (see our paper *Classifying spaces for sheaves of simplicial groupoids*, JPAA 1993) says that isomorphism classes of such fiber bundles over B are in 1 - 1 correspondance with homotopy classes of maps k over $K(\pi_1, 1)$. The above adjunction says these in turn are isomorphic to

$$[\tilde{B}, K(\tilde{\pi}, n+1)] \simeq H^{n+1}_{\pi_1}(\tilde{B}, \tilde{\pi})$$

So it is this equivariant cohomology that classifies fiber bundles over B with fiber $K(\pi, n)$.