2-bicategories of polynomials

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Background on lccc's

Recall given $f : X \to Y$ in $\mathcal{E}$ a category with pullbacks

$$\begin{tikzcd}
\mathcal{E}/X \arrow[rr, shift left=8, "\Sigma_f"] \arrow[rr, shift left=4, phantom, "\perp"] \arrow[rr, shift right=4, phantom, "\perp"] \arrow[rr, dotted, shift right=8, "\Pi_f"'] & & \mathcal{E}/Y \arrow[ll, "\Delta_f"]
\end{tikzcd}$$

When Δ_f has a further right adjoint, denoted Π_f, f is said to be **exponentiable**.

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Recall given $f : X \to Y$ in $\mathcal{E}$ a category with pullbacks

$$\begin{tikzcd}[column sep=huge] \mathcal{E}/X \arrow[r, shift left=6, "\Sigma_f"] \arrow[r, phantom, shift left=3, "\perp"] & \mathcal{E}/Y \arrow[l, "\Delta_f" description] \\ \end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge] \mathcal{E}/X \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, dotted, shift right=3, "\Pi_f"'] & \mathcal{E}/Y \end{tikzcd}$$

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$$\begin{tikzcd} \mathcal{E}/X \arrow[rr, bend left, "\Sigma_f"] \arrow[rr, phantom, shift left=2, "\perp"] \arrow[rr, phantom, shift right=2, "\perp"] \arrow[rr, dotted, bend right, "\Pi_f"'] & & \mathcal{E}/Y \arrow[ll, "\Delta_f"'] \end{tikzcd}$$

When Δ_f has a further right adjoint, denoted Π_f, f is said to be **exponentiable**. When $\mathcal{E}$ has finite limits and all its morphisms are exponentiable, $\mathcal{E}$ is said to be **locally cartesian closed**. Toposes are l.c.c but **CAT** is not.

Key notions

The **polynomial functor** $\mathcal{E}/X \to \mathcal{E}/Y$ associated to a polynomial

$$p \quad : \quad X \xleftarrow{p_1} A \xrightarrow{p_2} B \xrightarrow{p_3} Y$$

is the composite $\mathbf{P}(p) := \Sigma_{p_3}\Pi_{p_2}\Delta_{p_1}$.

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is the composite $\mathbf{P}(p) := \Sigma_{p_3}\Pi_{p_2}\Delta_{p_1}$. A morphism of polynomials is a diagram of the form

$$\begin{tikzcd} & A \arrow[dl, "p_1"'] \arrow[r, "p_2"] \arrow[d, "f_0"'] \arrow[dr, phantom, "pb"] & B \arrow[dr, "p_3"] \arrow[d, "f_1"] & \\ X & A' \arrow[l, "q_1"] \arrow[r, "q_2"'] & B' \arrow[r, "q_3"'] & Y \end{tikzcd}$$

and induces a cartesian transformation $\mathbf{P}(p) \to \mathbf{P}(q)$.

Monoid monad on **Set**

Example

$$
\begin{tikzcd}
 & \mathbb{N}_\bullet \times_{\mathbb{N}} \mathbb{N}^* \arrow[dl] \arrow[dd] \arrow[r] \arrow[ddr, phantom, "pb"] & \mathbb{N}^* \arrow[dd, "+"] \arrow[dr] & \\
1 & & & 1 \\
 & \mathbb{N}_\bullet \arrow[ul] \arrow[r] & \mathbb{N} \arrow[ur] &
\end{tikzcd}
$$

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1 & & & 1 \\
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\end{tikzcd}
$$

gives rise to the multiplication for the monoid monad.

Theorem

(Gambino and Kock 2009) Let $\mathcal{E}$ be locally cartesian closed.

1. *Objects of $\mathcal{E}$, polynomials over $\mathcal{E}$ and morphisms of polynomials form a bicategory $\mathbf{Poly}_{\mathcal{E}}$.*

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However the examples of polynomials we are interested in are in **CAT**. Also of interest are polynomials in **Top** – Bisson and Joyal, *The Dyer-Lashof Algebra in Bordism*, 1995.

Outline

1 Basic generalisation

Outline

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Generalisation to all categories with pullbacks

Theorem

Let $\mathcal{E}$ be a category with pullbacks.

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2. *The construction of polynomial functors from polynomials gives a homomorphism $\mathbf{P}_{\mathcal{E}} : \mathbf{Poly}_{\mathcal{E}} \to \mathbf{CAT}$.*

Composition of polynomials

$$\begin{tikzcd} \bullet & \bullet \arrow[l, "p_1"'] \arrow[r, "p_2"'] & \bullet \arrow[r, "p_3"'] & \bullet & \bullet \arrow[l, "q_1"'] \arrow[r, "q_2"'] & \bullet \arrow[r, "q_3"'] & \bullet \end{tikzcd}$$

Composition of polynomials

$$\begin{tikzcd}
& & & \bullet \arrow[dl] \arrow[dr] \arrow[d, phantom, "pb"] & & & \\
\bullet & \bullet \arrow[l, "p_1"'] \arrow[r, "p_2"'] & \bullet \arrow[r, "p_3"'] & \bullet & \bullet \arrow[l, "q_1"'] \arrow[r, "q_2"'] & \bullet \arrow[r, "q_3"'] & \bullet
\end{tikzcd}$$

Composition of polynomials

At this point one can consider the category of triples of morphisms (α, β, γ) as shown

$$
\begin{tikzcd}
 & & & \bullet \arrow[rr, dotted, "\beta"] \arrow[d, dotted, "\alpha"'] \arrow[drr, phantom, "pb"] & & \bullet \arrow[dd, dotted, "\gamma"] & \\
 & & & \bullet \arrow[dl] \arrow[dr, "f"] \arrow[d, phantom, "pb"] & & & \\
\bullet & \bullet \arrow[l, "p_1"'] \arrow[r, "p_2"'] & \bullet \arrow[r, "p_3"'] & \bullet & \bullet \arrow[l, "q_1"'] \arrow[r, "q_2"'] & \bullet \arrow[r, "q_3"'] & \bullet
\end{tikzcd}
$$

making the square with boundary $(f\alpha, q_2, \gamma, \beta)$ a pullback.

Composition of polynomials

The terminal such is the **distributivity pullback** of f along q_2.

$$
\begin{tikzcd}
 & & & \bullet \arrow[rr] \arrow[d] \arrow[drr, phantom, "dpb"] & & \bullet \arrow[dd, "\Pi_{q_2}(f)"] & \\
 & & & \bullet \arrow[dl] \arrow[dr, "f"] \arrow[d, phantom, "pb"] & & & \\
\bullet & \bullet \arrow[l, "p_1"'] \arrow[r, "p_2"'] & \bullet \arrow[r, "p_3"'] & \bullet & \bullet \arrow[l, "q_1"] \arrow[r, "q_2"'] & \bullet \arrow[r, "q_3"'] & \bullet
\end{tikzcd}
$$

Composition of polynomials

$$
\begin{tikzcd}
 & & \bullet \arrow[r] \arrow[ddl] \arrow[dr, phantom, "pb"] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[ddr] \arrow[dl, phantom, "dpb"] & & \\
 & & & \bullet \arrow[dl] \arrow[dr] \arrow[d, phantom, "pb"] & & & \\
\bullet & \bullet \arrow[l, "p_1"'] \arrow[r, "p_2"'] & \bullet \arrow[r, "p_3"'] & \bullet & \bullet \arrow[l, "q_1"'] \arrow[r, "q_2"'] & \bullet \arrow[r, "q_3"'] & \bullet
\end{tikzcd}
$$

Composition of polynomials

$$
\begin{tikzcd}
 & & & \bullet \arrow[rrr] \arrow[rrrrrr, bend left, "r_2"] \arrow[ddlll, swap, "r_1"] \arrow[ddl] \arrow[drrr, phantom, "pb"] & & & \bullet \arrow[rrr] \arrow[d] \arrow[drrr, phantom, "dpb"] & & & \bullet \arrow[ddr] \arrow[ddrrr, "r_3"] & & & \\
 & & & & & & \bullet \arrow[dll] \arrow[drr] \arrow[d, phantom, "pb"] & & & & & & \\
\bullet & & \bullet \arrow[ll, "p_1"] \arrow[rr, swap, "p_2"] & & \bullet \arrow[rr, swap, "p_3"] & & \bullet & & \bullet \arrow[ll, "q_1"] \arrow[rr, swap, "q_2"] & & \bullet \arrow[rr, swap, "q_3"] & & \bullet
\end{tikzcd}
$$

Lemma

(Composition/cancellation) Given

$$
\begin{tikzcd}
B_6 \arrow[r, "h_9"] \arrow[d, "h_8"'] \arrow[dr, phantom, "pb"] & B_4 \arrow[r, "h_6"] \arrow[d, "h_5"] \arrow[dddr, phantom, "pb"] & B_5 \arrow[ddd, "h_7"] \\
B_2 \arrow[r, "h_3"'] \arrow[d, "h_2"'] \arrow[ddr, phantom, "dpb"] & B_3 \arrow[dd, "h_4"] & \\
B \arrow[d, "h"'] & & \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}
$$

in any category with pullbacks, then the right-most pullback is a distributivity pullback around (g, h_4) iff the composite diagram is a distributivity pullback around (gf, h).

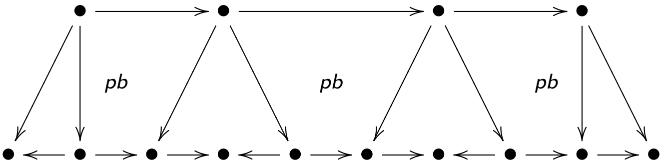
Lemma

(The cube lemma). Given

$$\begin{tikzcd}
A_1 \arrow[rrrr, "f_1"] \arrow[dr, "d_1"] \arrow[ddddd, "h_1"'] \arrow[ddr, phantom, "(1)"] & & & & B_1 \arrow[dl, "d_5"'] \arrow[ddddd, "k_1"] \\
& A_3 \arrow[dr, "d_2"] \arrow[ddd, "h_3"'] \arrow[ddr, phantom, "pb"] \arrow[rr, phantom, "(3)"] & & & \\
& & A_2 \arrow[r, "f_2"] \arrow[d, "h_2"'] \arrow[dr, phantom, "pb"] & B_2 \arrow[d, "k_2"] \arrow[dr, phantom, "(2)"] & \\
& & C_2 \arrow[r, "g_2"'] \arrow[dr, phantom, "dpb"] & D_2 & \\
& C_3 \arrow[ur, "d_4"] & & & \\
C_1 \arrow[ur, "d_3"] \arrow[rrrr, "g_1"'] & & & & D_1 \arrow[uul, "d_6"]
\end{tikzcd}$$

where (3) is a pullback. Then (1) and (2) are pullbacks iff (3) is a distributivity pullback.

Unbiased composition of polynomials



Enrichment over $\mathbf{CAT}_{\mathrm{pb}}$

The category $\mathbf{CAT}_{\mathrm{pb}}$ of categories with pullbacks and pullback preserving functors is cartesian closed. The internal hom $[X, Y]$ is the category of pullback preserving functors $X \to Y$ and cartesian transformations between them.

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A $\mathbf{CAT}_{\mathrm{pb}}$-*bicategory* is a bicategory $\mathcal{B}$ whose homs have pullbacks and whose compositions

$$\mathrm{comp}_{X,Y,Z} : \mathcal{B}(Y,Z) \times \mathcal{B}(X,Y) \to \mathcal{B}(X,Z)$$

preserve them. Categories enriched in $\mathbf{CAT}_{\mathrm{pb}}$ are exactly those $\mathbf{CAT}_{\mathrm{pb}}$-bicategories whose underlying bicategory is a 2-category.

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A *homomorphism* $F : \mathcal{B} \to \mathcal{C}$ of $\mathbf{CAT}_{\mathrm{pb}}$-bicategories is a homomorphism of their underlying bicategories whose hom functors preserve pullbacks.

Iterability of the theory

Theorem

Let $\mathcal{E}$ be a category with pullbacks. Then $\mathbf{Poly}_{\mathcal{E}}$ *is a* $\mathbf{CAT}_{\mathrm{pb}}$*-bicategory and*

$$\mathbf{P}_{\mathcal{E}} : \mathbf{Poly}_{\mathcal{E}} \to \mathbf{CAT}_{\mathrm{pb}}$$

is a homomorphism of $\mathbf{CAT}_{\mathrm{pb}}$*-bicategories.*

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is a homomorphism of $\mathbf{CAT}_{\mathrm{pb}}$*-bicategories.*

Since the homs of $\mathbf{Poly}_{\mathcal{E}}$ also have pullbacks we can apply the theorem to any of those homs in place of $\mathcal{E}$.

A **2-bicategory** is a bicategory $\mathcal{B}$

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3-categories $\subset$ 2-bicategories $\subset$ Tricategories

Theorem

Let $\mathcal{K}$ be a 2-category with pullbacks. Then $\mathbf{Poly}_{\mathcal{K}}$ *is a 2-bicategory and*

$$\mathbf{P}_{\mathcal{K}} : \mathbf{Poly}_{\mathcal{K}} \to \mathbf{2\text{-}CAT}$$

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is a homomorphism of 2-bicategories.

A **pseudo-monad** on an object X of a 2-bicategory $\mathcal{B}$, is a pseudo-monoid in the monoidal 2-category $\mathcal{B}(X, X)$.

2-topos examples

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$$1 \longleftarrow \mathbf{Set}_{\bullet} \xrightarrow{U} \mathbf{Set} \longrightarrow 1$$

is a biterminal object of $\mathcal{S}(1,1)$.

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is a biterminal object of $\mathcal{S}(1,1)$. Thus it carries a canonical polynomial pseudo-monad structure. The corresponding pseudo-monad on $\mathbf{CAT}$ is the Fam-construction.

More generally, replace $\mathbf{CAT}$ by a finitely complete $\mathcal{K}$ whose discrete opfibrations are exponentiable and U by a classifying discrete opfibration in $\mathcal{K}$.

Street's internal fibrations

Recall that the inclusion $\Delta \hookrightarrow \mathbf{CAT}$ is a cocategory object, and its canonical generators enjoy some lovely adjointnesses

$$
\begin{tikzcd}[column sep=huge]
{[0]} \arrow[r, shift left=6, "\delta_1"] \arrow[r, shift left=3, phantom, "\perp"] \arrow[r, shift right=3, phantom, "\perp"] \arrow[r, shift right=6, "\delta_0"'] & {[1]} \arrow[l, "\sigma_0"'] \arrow[r, shift left=12, "\delta_2"] \arrow[r, shift left=9, phantom, "\perp"] \arrow[r, "\delta_1"] \arrow[r, shift left=3, phantom, "\perp"] \arrow[r, shift right=3, phantom, "\perp"] \arrow[r, shift right=9, phantom, "\perp"] \arrow[r, shift right=12, "\delta_0"'] & {[2]} \arrow[l, shift right=6, "\sigma_1"'] \arrow[l, shift left=6, "\sigma_0"'] \arrow[r, shift left=18, "\delta_3"] \arrow[r, shift left=15, phantom, "\perp"] \arrow[r, shift left=9, phantom, "\perp"] \arrow[r, shift left=6, "\delta_2"] \arrow[r, shift left=3, phantom, "\perp"] \arrow[r, shift right=3, phantom, "\perp"] \arrow[r, shift right=6, "\delta_1"] \arrow[r, shift right=9, phantom, "\perp"] \arrow[r, shift right=15, phantom, "\perp"] \arrow[r, shift right=18, "\delta_0"'] & {[3]} \arrow[l, shift right=12, "\sigma_2"'] \arrow[l, "\sigma_1"'] \arrow[l, shift left=12, "\sigma_0"'] & \cdots\cdots
\end{tikzcd}
$$

Street's internal fibrations

Reinterpretting a little ...

$$
\begin{tikzcd}[column sep=large, row sep=large]
 & & {[0]} \arrow[dll, bend right, "t"'] \arrow[dl, bend right, "\delta_0 = t"'] \arrow[d, "t"] \arrow[dr, bend left, "t"] & & \cdots \\
{[0]} & {[1]} \arrow[l, "\sigma_0"'] \arrow[r] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & {[2]} \arrow[l, shift right=5, "\sigma_0"'] \arrow[l, shift left=5, "\sigma_1"] \arrow[r, shift left=5] \arrow[r, shift right=5] \arrow[r, phantom, shift left=7, "\perp"] \arrow[r, phantom, shift left=2.5, "\perp"] \arrow[r, phantom, shift right=2.5, "\perp"] \arrow[r, phantom, shift right=7, "\perp"] & {[3]} \arrow[l] \arrow[l, shift right=10, "\sigma_0"'] \arrow[l, shift left=10, "\sigma_2"] & \cdots\cdots \\
 & & {[0]} \arrow[ull, bend left, "b"] \arrow[ul, bend left, "\delta_1 = b"] \arrow[u, "b"'] \arrow[ur, bend right, "b"'] & & \cdots
\end{tikzcd}
$$

is a lax idempotent pseudo monad (on [1]) in the 2-bicategory $\mathbf{Cospan}_{\mathbf{CAT}}$.

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Let X be an object of a finitely complete 2-category $\mathcal{K}$.

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Cotensoring the previous slide with X gives a lax idempotent pseudo monad (on X) in the 2-bicategory $\mathbf{Span}_{\mathcal{K}}$, which sits inside $\mathbf{Poly}_{\mathcal{K}}$.

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The associated pseudo monad on $\mathcal{K}/X$ is the monad for fibrations.

Local right adjoints vs polynomial functors

A functor $F : \mathcal{A} \to \mathcal{B}$ is a **local right adjoint** when for all $X \in \mathcal{A}$ the induced functor

$$F_X : \mathcal{A}/X \to \mathcal{B}/FX$$

is a right adjoint. When $\mathcal{A}$ has 1, it suffices to check this for $X = 1$.

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Polynomial functors are l.r.a because for a polynomial p, the composite $\Pi_{p_2}\Delta_{p_1}$ may be identified with $\mathbf{P}_{\mathcal{E}}(p)_1$. Notice that the left adjoint to $\mathbf{P}_{\mathcal{E}}(p)_1$ is $\Sigma_{p_1}\Delta_{p_2}$ which itself preserves connected limits and thus in particular monos.

Local right adjoints vs polynomial functors

Example

The category monad T on **Gph** is l.r.a. but not polynomial over **Gph**.

Local right adjoints vs polynomial functors

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The category monad T on **Gph** is l.r.a. but not polynomial over **Gph**.The left adjoint $L_T : \mathbf{Gph}/T1 \to \mathbf{Gph}$ to T_1, applied to a labelled graph, replaces each edge labelled by n by a path of length n. In particular, the source and target of an edge labelled by 0 are identified, and so L_T does not preserve monos.

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Given *Polynomial functors and opetopes* – BJKM 2007, this is a little sad.

Polynomial from a l.r.a between presheaf categories

Given $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{D}}$ l.r.a one has

$$
\begin{tikzcd}
\widehat{\mathbb{C}} \arrow[r, shift right=1.5, "T_1"'] \arrow[r, phantom, "\perp"] & \widehat{\mathbb{D}}/T1 \arrow[l, shift right=1.5, "L_T"'] \arrow[r, "\Sigma_{T1}"] & \widehat{\mathbb{D}} \\
& y_{\mathbb{D}}/T1 \arrow[ul, "E_T"] \arrow[u, hook] &
\end{tikzcd}
$$

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$$\begin{tikzcd}
\widehat{\mathbb{C}} \arrow[r, shift right=1.5, "T_1"'] & \widehat{\mathbb{D}}/T1 \arrow[l, shift right=1.5, "L_T"'] \arrow[l, phantom, "\perp"] \arrow[r, "\Sigma_{T1}"] & \widehat{\mathbb{D}} \\
& y_{\mathbb{D}}/T1 \arrow[ul, "E_T"] \arrow[u, hook] &
\end{tikzcd}$$

From which one produces the polynomial $p_T : \mathbb{C} \to \mathbb{D}$

$$\mathbb{C} \xleftarrow{p_{T,1}} y_{\mathbb{C}}/E_T \xrightarrow{p_{T,2}} y_{\mathbb{D}}/T1 \xrightarrow{p_{T,3}} \mathbb{D}$$

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2. *By applying* $\mathbf{P}(p_T)$ *to discrete fibrations.*