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Codensity and double-dualization monads

It is known since 1970's that the codensity monad of the embedding of finite sets into Set is the ultrafilter monad. Leinster proved in [1] that the full embedding of finite-dimensional vector spaces into K-Vec has the codensity monad given by the double-dualization monad $(-)^{**}$. And he asked for generalizations covering the two examples above. We present a solution working in categories $\mathcal{K}$ that are monoidal closed and have a strong cogenerator D. The functor $(-)^* = [-, D]$ is left adjoint to its dual, and the resulting monad $(-)^{**}$ is called the double-dualization monad.

Example. Varieties of algebras have a 'natural' tensor product, representing bimorphisms. Monoidal closedness means precisely that the variety (or, equivalently, its monad) is commutative, see [2]. Analogously, varieties of ordered algebras, presented by operations and inequations, are monoidal closed iff they are commutative.

Definition. By the **finite double-dualization monad** is meant the largest submonad of $(-)^{**}$ whose unit has invertible components at all finitely presentable objects.

Theorem. *Let $\mathcal{K}$ be a commutative variety of (possibly ordered) algebras. Let D be a strong cogenerator with D^n finitely presentable for all $n \in N$. Then the finite double-dualization monad is the codensity monad of the full embedding of all finitely presentable objects into $\mathcal{K}$.*

Examples. (a) K is a strong cogenerator of K-Vec. Since for finitely-dimensional spaces the unit $\eta_A : A \to A^{**}$ is invertible, we obtain Leinster's result that the codensity monad is all of $(-)^{**}$.
(b) The category $\mathcal{JSL}$ of join semilattices has the two-element chain as a strong cogenerator. Again, finite semilattices have invertible units, hence, the codensity monad of their embedding is also $(-)^{**}$.
(c) For $\mathcal{S}et$ the two-element set as a cogenerator yields $X^* = \mathcal{P}X$. The finite double-dualization monad is the ultrafilter monad.
(d) Analogously for $\mathcal{P}os$: take the two-element chain as a strong cogenerator. Then X^* is the poset $\mathcal{P}^u X$ of all up-sets of X, ordered by the dual of inclusion. The finite double-dualization monad is the prime-filter monad on $\mathcal{P}os$.

Remark. We further study codensity monads of set functors. Every accessible functors posseses a codensity monad. The converse does not hold:

Example. (1) For the power-set functor $\mathcal{P}$ the codensity monad assigns to X the product $\prod_{Y \subseteq X} \mathcal{P}Y$.

[*] Joint work with Lurdes Sousa.

(2) For the subfunctor $\mathcal{P}_0$ of all nonempty subsets the codensity monad is $\mathcal{P}_0$ itself.
(3) In contrast, the following modification $\mathcal{P}'$ of $\mathcal{P}$ does not posses a codensity monad: on objects $\mathcal{P}'X = \mathcal{P}X$, on morphisms $f : X \to Y$, for every $M \subseteq X$ put $\mathcal{P}'f(M) = \mathcal{P}f(M)$ in case f/M is monic, else $\emptyset$.

References:

[1] T. Leinster, Codensity and the ultrafilter monad. *Theory and Applications of Categories* 28 (2013), 332–370.

[2] B. Banaschewski and N. Nelson, Tensor products and bimorphisms. *Canad. Math. Bull.* 19 (1976), 385–402.