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On flat 2-functors

The main theorem of the theory of flat functors ([1], [4]) states that $A \xrightarrow{P} \mathcal{E}ns$ is flat if and only if P is a filtered colimit of representable functors, i.e. there is a filtered category I and a diagram $I \xrightarrow{X} A$ such that P is the colimit of the composition $I^{op} \xrightarrow{X} A^{op} \xrightarrow{h} Hom(A, \mathcal{E}ns)$ where h is the Yoneda embedding. For an arbitrary *base* category $\mathcal{V}$ instead of $\mathcal{E}ns$, Kelly ([3]) has developed a theory of flat $\mathcal{V}$-enriched functors $A \xrightarrow{P} \mathcal{V}$, but there is no known generalization of the theorem above for any $\mathcal{V}$ other than $\mathcal{E}ns$.

In [2] we have established a 2-dimensional version of this theorem, i.e. for a 2-functor $\mathcal{A} \xrightarrow{P} \mathcal{C}at$, where $\mathcal{A}$ is a 2-category and $\mathcal{C}at$ is the 2-category of categories. As it is usually the case for 2-categories, the $\mathcal{C}at$-enriched notions are not adequate for most purposes and the *relaxed* bi and pseudo notions are the important ones.

We define a 2-functor $\mathcal{A} \xrightarrow{P} \mathcal{C}at$ to be *flat* when its *left bi-Kan extension* $\mathcal{H}om_s(\mathcal{A}^{op}, \mathcal{C}at) \xrightarrow{P^*} \mathcal{C}at$ along the Yoneda 2-functor $\mathcal{A} \xrightarrow{h} \mathcal{H}om_s(\mathcal{A}^{op}, \mathcal{C}at)$ is *left exact*. $\mathcal{H}om_s(\mathcal{A}^{op}, \mathcal{C}at)$ denotes the 2-category of 2-functors, 2-natural transformations and modifications. By left bi-Kan extension we understand the bi-universal pseudonatural transformation $P \Longrightarrow P^*h$, and by left exact we understand preservation of finite weighted bilimits. Let $(\mathcal{A},\ \Sigma)$ be a pair where $\mathcal{A}$ is a 2-category and Σ a distinguished 1-subcategory. A σ-cone for a 2-functor $\mathcal{A} \xrightarrow{F} \mathcal{B}$ is a lax cone such that the 2-cells corresponding to the distinguished arrows are invertible. The *σ-limit* of F is a universal σ-cone (characterized up to isomorphism). We introduce a notion of 2-filteredness of $\mathcal{A}$ with respect to Σ, which we call *σ-filtered*. Our main result states the following:

A 2-functor $\mathcal{A} \xrightarrow{P} \mathcal{C}at$ is flat if and only if there is a σ-filtered pair $(\mathcal{I}^{op},\ \Sigma)$ and a 2-diagram $\mathcal{I} \xrightarrow{X} \mathcal{A}$ such that P is pseudo-equivalent to the σ-bicolimit of the composition $\mathcal{I}^{op} \xrightarrow{X} \mathcal{A}^{op} \xrightarrow{h} \mathcal{H}om_s(\mathcal{A}, \mathcal{C}at)$. As in the 1-dimensional case, X can be chosen as the 2-fibration associated to P.

References:

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