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Duality theory, convergence, and enriched categories

The principal aim of this talk is to combine the three keywords of the title in a suitable way. It is probably impossible to talk about duality theory without mentioning Stone's famous duality results for Boolean algebras and distributive lattices, which motivated many other duality results, typically involving some kind of lattices. Our first goal is to develop a systematic method, based on enriched category theory, for extending these duality theorems to categories including all compact Hausdorff spaces. Keeping in mind that ordered sets can be viewed as categories enriched in the two-element quantale, our thesis is that *the passage from the two-element space to the compact Hausdorff space* $[0,1]$ (*a cogenerator of the category of compact Hausdorff spaces*) *on one side of the duality should be matched by a move from ordered structures to categories enriched in* $[0,1]$ *on the other side.* Accordingly, we present duality theory for ordered compact Hausdorff spaces and monoids of categories enriched in the quantale $[0,1]$ with finite weighted colimits. One should think of these monoids as "$[0,1]$-enriched lattices".

However, doing so is somehow inconsequential, as we still consider *ordered* compact Hausdorff spaces. Our next step aims at an extension of these results to compact Hausdorff spaces equipped with a quantale-enriched category structure, which constitute a generalisation of Nachbin's ordered spaces (see [6, 7]) and are closely related to Hermida's representable multi-categories [4]. Arguably, these spaces are best studied within the framework of "quantale-enriched topological spaces"; that is, lax algebras for the ultrafilter monad *à la* Barr's description of topological spaces as relational algebras [1]. We use this opportunity to recall the setting of monad-quantale enriched categories [5] and in particular the important notion of distributor. We sharpen some results on Cauchy-completeness presented earlier, and give a more systematic study of enriched compact Hausdorff spaces. If time permits, we will also consider the case of an enrichment in a symmetric monoidal closed category (see [2]).

Finally, already Halmos [3] observed that it is often beneficial to study duality theory "at a slightly more general level than might appear relevant at first sight", and proved that the category of Boolean spaces and Boolean *relations* is dually equivalent to the category of Boolean algebras and maps preserving finite suprema

$$\mathsf{BooRel} \simeq \mathsf{FinSup}_{\mathrm{boo}}^{\mathrm{op}};$$

here BooRel is actually the Kleisli category of the Vietoris monad, and the latter category we describe as the full subcategory of the category of finitely complete ordered sets defined by Boolean algebras. Using again the theory of monad-quantale enriched categories, we introduce and study enriched versions of the classical Vietoris monad. With these tools at our disposal, we develop duality theory for $[0,1]$-enriched compact Hausdorff spaces and distributors on one side, and categories enriched in the quantale $[0,1]$ with finite weighted colimits on the other side. These results entail the duality results mentioned before; surprisingly or not, the general theory seems to work better in this setting. We also use these results to show that the dual of the category of partially ordered compact Hausdorff spaces is a $\aleph_1$-ary quasivariety and give a partial description of its algebraic theory, which is sufficient to identify also the dual of the category of Vietoris coalgebras as a $\aleph_1$-ary quasivariety.

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