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A synthetic theory of ∞-categories in homotopy type theory

One of the observations that launched homotopy type theory is that the rule of identity-elimination in Martin-Löf's identity types automatically generates the structure of an ∞-groupoid. In this way, homotopy type theory can be viewed as a "synthetic theory of ∞-groupoids." It is natural to ask whether there is a similar *directed* type theory that describes a "synthetic theory of $(\infty, 1)$-categories," but on account of a number of technical obstructions, this has long proven elusive.

In this talk, we propose foundations for a synthetic theory of $(\infty, 1)$-categories in homotopy type theory [1] motivated by the model of homotopy type theory in the category of Reedy fibrant simplicial spaces [2], which contains as a full subcategory the ∞-cosmos of Rezk spaces (aka complete Segal spaces) [3], a well-known model of $(\infty, 1)$-categories whose category theory can be developed synthetically [4]. We introduce simplices and cofibrations into homotopy type theory to probe the internal categorical structure of types, and define *Segal types*, in which binary composites exist uniquely up to homotopy, and *Rezk types*, in which the categorical isomorphisms are equivalent to the type-theoretic identities — a "local univalence" condition. In the simplicial spaces model these correspond exactly to the Segal and Rezk spaces. We then demonstrate that these simple definitions suffice to develop the synthetic theory of $(\infty, 1)$-categories. So far this includes functors, natural transformations, co- and contravariant type families with discrete (∞-groupoid) fibers, a "dependent" Yoneda lemma that looks like "directed identity-elimination," and the theory of coherent adjunctions closely resembling [5].

References:

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[4] Emily Riehl and Dominic Verity, ∞-category theory from scratch, arXiv:1608.05314 (2015), 1–53.

[5] Emily Riehl and Dominic Verity, Homotopy coherent adjunctions and the formal theory of monads, *Advances in Mathematics* 286(2) (2016) 802–888.

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