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Categorical-algebraic methods in group cohomology

In the article [8], Janelidze introduced the concept of a *double central extension* in order to analyse the Hopf formula for the third integral homology of a group [2]. Later it turned out that this "double extension" viewpoint on group homology may be extended to higher degrees, and at the same time generalised to the framework of semi-abelian categories [5]. Indeed, categorical Galois theory gives rise to the concept of an *n-fold central extension* ($n \geq 1$), which is such that the higher Hopf formulae of [2, 3], suitably reinterpreted in terms of these higher central extensions, give an explicit description of the derived functors of any reflector from a semi-abelian variety to one of its subvarieties. In the particular case of the abelianisation reflection from the category of groups to the category of abelian groups, the Hopf formulae for integral group homology are thus regained.

Central extensions do however also appear in group *co*homology, in the interpretation of the second cohomology group with coefficients in a trivial $\mathbb{Z}$-module A, which is one of the derived functors of the functor $\mathrm{Hom}(-, A)$. This result extends to semi-abelian categories [7] and to non-trivial coefficients (via the concept of a torsor [1]). On the other hand, in the abelian case there is Yoneda's classical interpretation of these derived functors via classes of exact sequences of a certain fixed length [10]. In Barr-exact categories, the higher-dimensional torsors of [4] play essentially the same role.

The aim of this talk is to explain how, in a semi-abelian context, these two developments are related. Through an equivalence between higher torsors (with trivial coefficients) and higher central extensions we obtain a duality, in a certain sense, between homology and cohomology [9, 6]. Even in the case of groups this viewpoint is new, but it is automatically valid as well for other non-abelian algebraic structures such as Lie algebras, crossed modules, associative algebras, and so on.

In its most general version, the theory depends on some non-trivial recent developments in categorical algebra. Part of the talk focuses on these categorical-algebraic aspects: how questions in homological algebra naturally lead to categorical conditions and results. The need for further development of categorical algebra becomes particularly apparent in the case of cohomology with non-trivial coefficients. This case is much more complicated, because here the techniques of categorical Galois theory are no longer available.

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