Andrea Montoli *
Università degli Studi di Milano

On the categorical behaviour of ordered groups

The aim of this talk is to explore the categorical-algebraic properties of the category **OrdGrp** of preordered groups and monotone group homomorphisms [2].

The split short five lemma does not hold in **OrdGrp**, so this category is not Bourn-protomodular, actually it is not even a Mal'tsev category. In fact, an exploration of the structure of split extensions in this category reveals a big variety of preordered structures on the domain of a split extension, and this variety causes the failure of strong categorical-algebraic properties.

However, **OrdGrp** is isomorphic to the category whose objects are pairs (G, M), where G is a group and M is a submonoid of G closed under conjugation, and whose morphisms are group homomorphisms that restrict to the submonoids. Thanks to this fact, we can see that the categorical-algebraic behaviour of **OrdGrp** is very similar to the one of the category **Mon** of monoids.

In particular, two recent approaches to a relativization of the notion of protomodularity appear to give meaningful information about the category **OrdGrp**. The two approaches are the objectwise one introduced in [3], which aims to identify some "good" objects in a category with weak algebraic properties, and the relative protomodular one introduced in [1], according to which the categorical-algebraic properties of a category are studied relatively to a suitable class of (split) epimorphisms.

Both approaches allow to identify a *protomodular core* inside a category **C**, namely a full subcategory which is protomodular and retains all the protomodular-like properties that could be identified in the bigger category **C**. If **C** = **OrdGrp**, both approaches determine the same protomodular core, which is the full subcategory of **OrdGrp** whose objects are the preordered groups whose preorder relation is symmetric. In particular, if we restrict our attention to strictly ordered groups (i.e. those preordered groups whose preorder relation is antisymmetric), the protomodular core is the full subcategory of discrete groups.

References:

[1] D. Bourn, N. Martins-Ferreira, A. Montoli, M. Sobral, Monoids and pointed S-protomodular categories, *Homology, Homotopy and Applications* 18 n.1 (2016) 151–172.

[2] M.M. Clementino, N. Martins-Ferreira, A. Montoli, On the categorical behaviour of ordered groups, in preparation.

[3] A. Montoli, D. Rodelo, T. Van der Linden, Two characterisations of groups amongst monoids, *J. Pure Appl. Algebra* 222 (2018) 747–777.

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