Paolo Perrone *
Max Planck Institute for Mathematics in the Sciences
Leipzig, Germany

On the operational meaning of the bar construction, with an application to probability

For each algebra of a monad, the *bar construction* [1, VII.6] gives a simplicial object. We extend the interpretation of monads in terms of formal operations to the whole bar construction. Consider two formal expressions, and suppose that one can be obtained from the other by "partial evaluation", using the rules specified by the monad. For example, "2+3" is a partial evaluation of "2+2+1". We then draw a 1-cell between them. The bar construction is a simplicial object whose points correspond to formal expressions, whose 1-cells correspond to "simplification rules", and whose higher order simplices correspond to rules between rules (for example, to define composite rules). We have some evidence suggesting that in many concrete categories the induced simplicial set may be a quasi-category. For example, the bar construction for a cartesian monad on sets is always the nerve of an ordinary category.

This construction has a powerful application to probability. Probability monads, like the Giry monad, encode "expectations" as their formal operations. Partial evaluations, in this case, are tightly linked to what probabilists call *conditional expectation*, and to how it is used in probability, for example to define martingales.

As an example we focus on a particular probability monad, the Kantorovich monad on the category of complete metric spaces [2, 3]. For each algebra, the (0,1)-truncation of its bar construction is a closed partial order, the opposite of the so-called *convex* or *Choquet order*, widely used in economics, convex analysis, and operator theory [4, 5]. As this truncation is already transitive and closed, we can equivalently obtain it as a *lax codescent object* in the sense of Lack [6], in a 2-category of directed spaces.

References:

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