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Temporal Type Theory:
A topos-theoretic approach to systems and behavior

A well-functioning logic of behavior is indispensable for studying interactions among real-world systems and processes, for which typical mathematical modeling frameworks range from that of dynamical systems to that of Turing machines. It appears that any behavior—e.g. a trajectory of an ordinary differential equation or an execution of a computer program—takes place over a duration of "time". Thus in [1] we propose the theory of real durations, i.e. translation-invariant intervals, as a formal system for studying behavior.

In this talk, I will recall the topological space $\mathbb{IR}$, called the *interval domain* [2], and discuss how its sheaves—objects in $\mathsf{Shv}(\mathbb{IR})$—can be conceptualized as types of behavior in the context of time. There is a quotient topos $\mathsf{Shv}(\mathbb{IR}) \to \mathsf{Shv}(\mathbb{IR}_{/\triangleright})$, defined in terms of the translation action $\triangleright$ on $\mathbb{IR}$ by the group $\mathbb{R}$ of real numbers, and we refer to $\mathcal{B} := \mathsf{Shv}(\mathbb{IR}_{/\triangleright})$ as the topos of *behavior types*. Using a special behavior type, which we denote $\mathtt{Time} \in \mathcal{B}$, one may recover $\mathsf{Shv}(\mathbb{IR}) \cong \mathcal{B}/\mathtt{Time}$ as a slice topos of $\mathcal{B}$.

I will explain the relationship between $\mathcal{B}$ and the topos studied by Lawvere in [4] and others (e.g. [5]), also in the context of abstract dynamical systems. I will also briefly discuss how to use the internal language of the topos $\mathcal{B}$ to describe the behaviors of both ODEs and state machines, in order to indicate how these two typical—but very different—behavior-modeling frameworks can interact within a single logical formalism, which we call *higher order temporal logic*.

References:

[1] Schultz, P. and Spivak, D.I., 2017. *Temporal Type Theory: A topos-theoretic approach to systems and behavior*. Birkaüser Mathematics (*to appear*).

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[3] Johnstone, P. and Joyal, A., 1982. Continuous categories and exponentiable toposes. *Journal of Pure and Applied Algebra*, 25(3), pp. 255–296.

[4] Lawvere, F.W., 1986. State categories and response functors. *preprint.*

[5] Spivak, D.I., Vasilakopoulou, C. and Schultz, P., 2016. Dynamical systems and sheaves. *preprint* arXiv:1609.08086.

*Joint work with Patrick Schultz.