A simplicial groupoid for plethysm*

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Plethystic substitution

Substitution operation in the ring of power series in infinitely many variables,

$$G(x_1, x_2, \dots) \circledast F(x_1, x_2, \dots) = G(F_1, F_2, \dots), \quad \text{where}$$

$$F_k(x_1, x_2, \dots) = F(x_k, x_{2k}, \dots).$$

- Pólya (1937): unlabelled enumeration. Given a species

$$F : \mathbb{B} \longrightarrow Set,$$

its cycle index series is a power series $Z_F(x_1, x_2, \dots)$. It satisfies

$$Z_{F \circ G} = Z_F \circledast Z_G$$

- Littlewood (1944): representation theory of GL. The character of the composition of polynomial representations is the plethysm of their characters.

Introduction

- Nava–Rota (1985): combinatorial interpretation of plethystic substitution based on partitions.

Goal

Recover $\circledast$ from the coproduct of $\mathbb{Q}_{\pi_0 T_1\mathbf{S}}$ for an explicit simplicial groupoid $T\mathbf{S} \colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Grpd}$.

$$T\mathbf{S} \colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Grpd} \underset{\text{bialgebra}}{\overset{\text{incidence}}{\rightsquigarrow}} \mathbf{Grpd}_{/T_1\mathbf{S}} \underset{\text{cardinality}}{\overset{\text{homotopy}}{\rightsquigarrow}} \mathbb{Q}_{\pi_0 T_1\mathbf{S}}$$

Contents

Faà di Bruno

One-variable power series

$$F(x) = \sum_{n=1} f_n \frac{x^n}{n!} \in \mathbb{Q}[[x]]$$

$$G(x) = \sum_{n=1} g_n \frac{x^n}{n!} \in \mathbb{Q}[[x]]$$

Faà di Bruno bialgebra $\mathcal{F}$

Free algebra $\mathbb{Q}[A_1, A_2, \ldots]$,

$$\begin{aligned} A_n \colon \mathbb{Q}[[x]] &\longrightarrow \mathbb{Q} \\ F &\longmapsto f_n, \end{aligned}$$

with coproduct

$$\Delta(A_n)(F \otimes G) = A_n(G \circ F)$$

Bell polynomials $B_{n,k}$

$$\Delta(A_n) = \sum_{k=1}^{n} B_{n,k}(A_1, A_2, \ldots) \otimes A_k$$

$B_{n,k}$ counts the number of surjections

$$n \twoheadrightarrow k \twoheadrightarrow 1$$

up to $k \xrightarrow{\sim} k$.

Example

$$B_{6,2} = 6A_1A_5 + 15A_2A_4 + 10A_3A_3$$

Faà di Bruno

Theorem (Joyal,1981)

$\mathcal{F}$ is isomorphic to the homotopy cardinality of the incidence bialgebra of the fat nerve of the category of finite sets and surjections

$$N\mathbf{S} \colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Grpd}.$$

This isomorphisms takes A_n to $n \twoheadrightarrow 1$, and $A_{n_1} \cdots A_{n_\ell}$ to $(n = n_1 + \cdots + n_\ell \twoheadrightarrow \ell)$. The coproduct is given by

$$\Delta(n \twoheadrightarrow \ell) = \sum_{n \twoheadrightarrow k \twoheadrightarrow \ell} (n \twoheadrightarrow k) \otimes (k \twoheadrightarrow \ell).$$

Remark

$$(n \twoheadrightarrow \ell) \xleftarrow{\ d_1\ } (n \twoheadrightarrow k \twoheadrightarrow \ell) \xrightarrow{(d_2, d_0)} (n \twoheadrightarrow k, k \twoheadrightarrow \ell)$$

Segal groupoids

A simplicial groupoid $X\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Grpd}$ is **Segal** if for all $n > 0$

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{d_0} & X_n \\ {\scriptstyle d_{n+1}}\downarrow & \lrcorner & \downarrow{\scriptstyle d_n} \\ X_n & \xrightarrow[d_0]{} & X_{n-1}. \end{array}$$

Example

The fat nerve of a category.

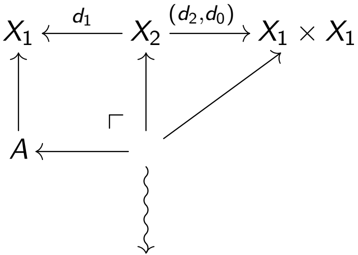
Remark

There is an up to equivalence "composition" given by

$$X_1 \times_{X_0} X_1 \xleftarrow{\sim} X_2 \xrightarrow{d_1} X_1,$$

which is actual composition when X is the fat nerve of a category.

Incidence bialgebras



$$\begin{array}{rcl} \Delta\colon \mathbf{Grpd}_{/X_1} & \longrightarrow & \mathbf{Grpd}_{/X_1\times X_1} \\ A \xrightarrow{s} X_1 & \longmapsto & (d_2, d_0)_! \circ d_1^*(s) \end{array}$$

In a similar way we obtain a functor

$$\epsilon\colon \mathbf{Grpd}_{/X_1} \longrightarrow \mathbf{Grpd}.$$

Theorem (Gálvez, Kock, Tonks)

If X is a Segal space the functors Δ and ϵ are respectively coassociative and counital, up to homotopy.

Incidence bialgebras

Definition

The slice groupoid $\mathbf{Grpd}_{/X_1}$ together with Δ and ϵ is the **incidence coalgebra** of X.

CULF monoidal structure

Product $X_n \times X_n \to X_n$ compatible with face and degeneracy maps and such that

$$\begin{array}{ccc} X_n \times X_n & \xrightarrow{g \times g} & X_1 \times X_1 \\ \downarrow & \lrcorner & \downarrow \\ X_n & \xrightarrow[g]{} & X_1, \end{array}$$

with g induced by the endpoint preserving map $[1] \to [n]$. Most of times in combinatorics the monoidal structure is disjoint union.

Incidence bialgebra

If X is CULF monoidal the incidence coalgebra becomes a bialgebra.

Homotopy cardinality

Homotopy cardinality of a groupoid

$$|\cdot|\colon \mathbf{Grpd} \longrightarrow \mathbb{Q}, \qquad |A| := \sum_{a\in\pi_0 A} \frac{1}{|\operatorname{Aut}(a)|} \in \mathbb{Q}$$

Homotopy cardinality of a finite map of groupoids $A \xrightarrow{p} B$

$$|\cdot|\colon \mathbf{Grpd}_{/B} \longrightarrow \mathbb{Q}_{\pi_0 B}, \qquad |p| := \sum_{b\in\pi_0 B} \frac{|A_b|}{|\operatorname{Aut}(b)|}\delta_b \in \mathbb{Q}_{\pi_0 B},$$

where A_b is homotopy fibre.

Remark

$$|1 \xrightarrow{\ulcorner b \urcorner} B| = \frac{|1_b|}{|\operatorname{Aut}(b)|}\delta_b = \delta_b$$

Homotopy cardinality

The homotopy cardinality of the incidence bialgebra of X gives a bialgebra structure on $\mathbb{Q}_{\pi_0 X_1}$.

$$\begin{array}{ccc}\Delta\colon \mathbf{Grpd}_{/X_1} \longrightarrow \mathbf{Grpd}_{/X_1 \times X_1} & & \epsilon\colon \mathbf{Grpd}_{/X_1} \longrightarrow \mathbf{Grpd} \\ \rightsquigarrow & & \rightsquigarrow \\ \Delta\colon \mathbb{Q}_{\pi_0 X_1} \longrightarrow \mathbb{Q}_{\pi_0 X_1} \otimes \mathbb{Q}_{\pi_0 X_1} & & \epsilon\colon \mathbb{Q}_{\pi_0 X_1} \longrightarrow \mathbb{Q}\end{array}$$

Plethystic substitution

Notation

- $\lambda = (\lambda_1, \lambda_2, \dots)$, nonzero infinite vector of natural numbers with finite number of nonzero entries,
- $\mathrm{aut}(\lambda) = 1!^{\lambda_1}\lambda_1! \cdot 2!^{\lambda_2}\lambda_2! \cdots$,
- $\mathbf{x}^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots$.

*n*th Verschiebung operator

Shifts the kth entry λ_k of λ to the nkth position. For example

$$V^2(5, 9, 2, 0 \dots) = (0, 5, 0, 9, 0, 2, 0 \dots).$$

Plethystic substitution

Remark

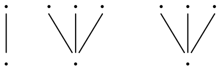
i) λ represents the isomorphism class of a surjection of finite sets

$$a \twoheadrightarrow b$$

with λ_k fibers of size k.

Example

$(1, 0, 2)$ corresponds to



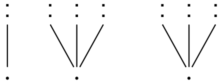
ii) $\mathrm{aut}(\lambda) = |\,\mathrm{Aut}(a \twoheadrightarrow b)|$

$$\begin{array}{ccc} a & \xrightarrow{\sim} & a \\ \downarrow & & \downarrow \\ b & \xrightarrow{\sim} & b \end{array}$$

iii) $V^n\lambda$ is the class of $n \times a \twoheadrightarrow b$

Example

$V^2(1, 0, 2) = (0, 1, 0, 0, 0, 2)$ corresponds to



Plethystic substitution

Infinitely many variables power series

$$F(\mathbf{x}) = \sum_{\mu} f_\mu \frac{\mathbf{x}^\mu}{\mathrm{aut}(\mu)} \in \mathbb{Q}[[\mathbf{x}]], \qquad G(\mathbf{x}) = \sum_{\lambda} g_\lambda \frac{\mathbf{x}^\lambda}{\mathrm{aut}(\lambda)} \in \mathbb{Q}[[\mathbf{x}]]$$

Plethystic substitution

$$(G \circledast F)(\mathbf{x}) = G(F_1, F_2, \dots), \text{ with}$$

$$F_k(x_1, x_2, \dots) = F(x_k, x_{2k}, \dots) = \sum_{\mu} f_\mu \frac{\mathbf{x}^{V^k \mu}}{\mathrm{aut}(\mu)}.$$

Plethystic substitution

Plethystic bialgebra $\mathcal{P}$

Free algebra $\mathbb{Q}[\{A_\lambda\}_\lambda]$,

$$\begin{aligned} A_\sigma : \mathbb{Q}[[\mathbf{x}]] &\longrightarrow \mathbb{Q} \\ F &\longmapsto f_\sigma, \end{aligned}$$

with coproduct

$$\Delta(A_\sigma)(F \otimes G) = A_\sigma(G \circledast F)$$

Polynomials $P_{\sigma,\lambda}(\{A_\mu\})$

$$\Delta(A_\sigma) = \sum_\lambda P_{\sigma,\lambda}(\{A_\mu\}) \otimes A_\lambda$$

What does $P_{\sigma,\lambda}$ count?

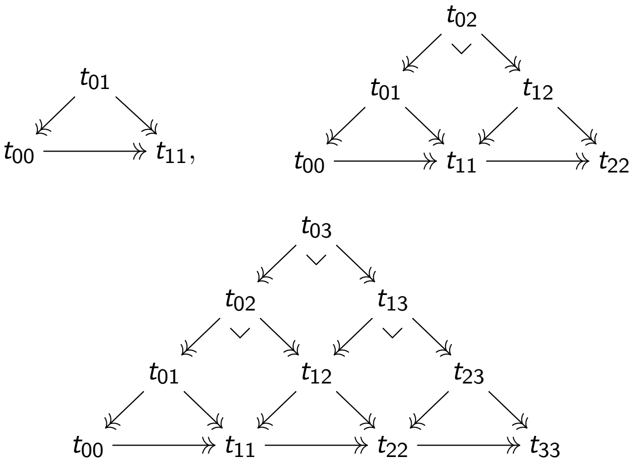
Example

$$P_{(0,0,0,1,0,2),(1,2)} = \frac{6!^2 2!4!}{4!3!^2 2!^2 2!} A_{(0,0,0,1)} A^2_{(0,0,1)} + \frac{6!^2 2!4!}{6!3!2!2!^2 2!} 2A_{(0,0,0,0,0,1)} A_{(0,0,1)} A_{(0,1)}$$

$$(0,0,0,1,0,2) = V^1(0,0,0,0,0,1) + V^2(0,0,1) + V^2(0,1)$$

$$(0,0,0,1,0,2) = V^1(0,0,0,1) + V^2(0,0,1) + V^2(0,0,1)$$

The simplicial groupoid $T\mathbf{S}\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Grpd}$

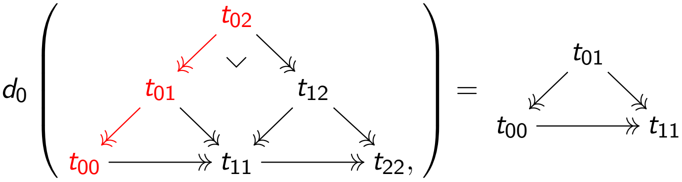


- t_{ij} are finite sets,
- $\twoheadrightarrow$ are surjections,
- every square is a pullback of finite sets.

The simplicial groupoid $T\mathbf{S}\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Grpd}$

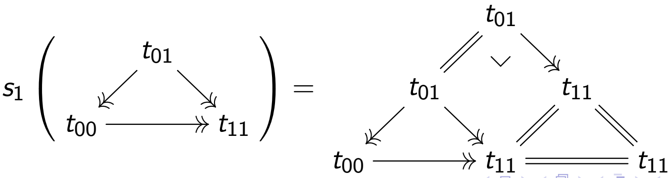
Face maps

d_i removes all the elements containing an i index:



Degeneracy maps

s_i repeats all the elements containing an i index:



The simplicial groupoid $T\mathbf{S}\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Grpd}$

Face maps

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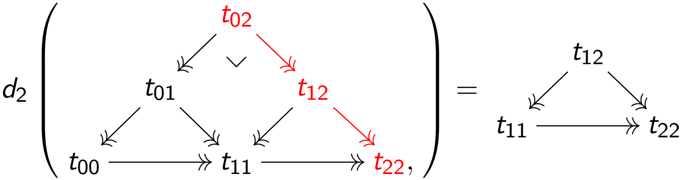

Degeneracy maps

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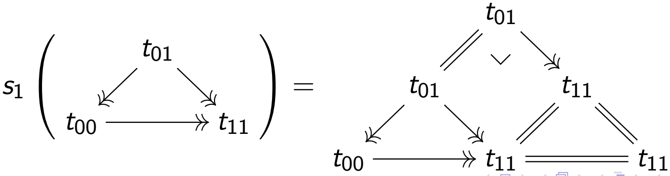

The simplicial groupoid $T\mathbf{S} \colon \Delta^{op} \longrightarrow \mathbf{Grpd}$

Face maps

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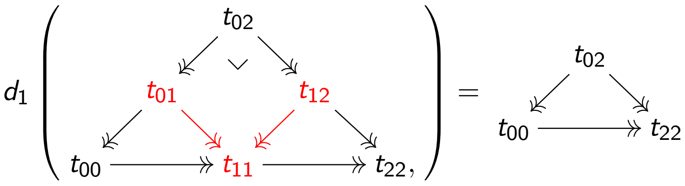

Degeneracy maps

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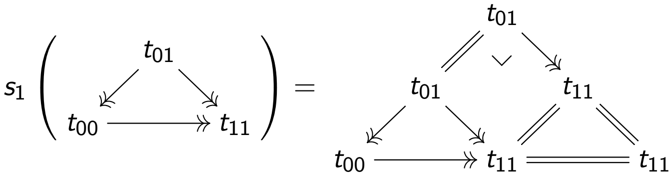

The simplicial groupoid $T\mathbf{S}\colon \Delta^{op} \longrightarrow \mathbf{Grpd}$

Face maps

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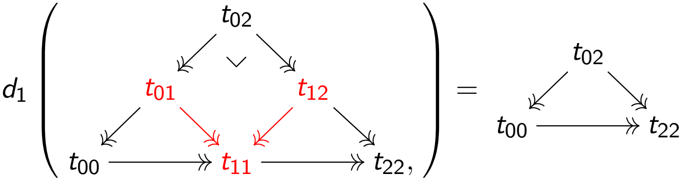

Degeneracy maps

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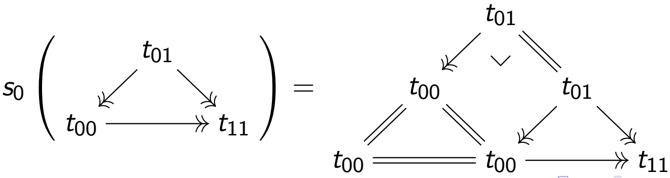
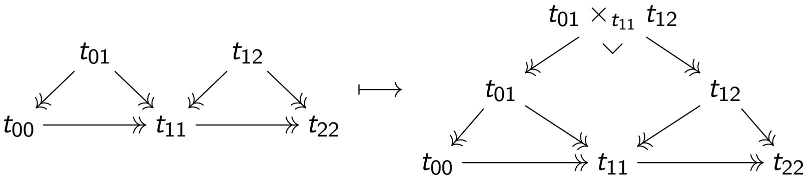


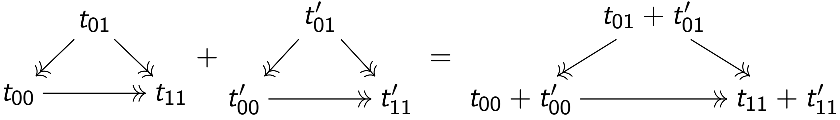
The simplicial groupoid $T\mathbf{S}\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Grpd}$

Proposition. *$T\mathbf{S}$ is a Segal groupoid.*

Equivalence $T_1\mathbf{S} \times_{T_0\mathbf{S}} T_1\mathbf{S} \xrightarrow{\sim} T_2\mathbf{S}$:



Proposition. *$T\mathbf{S}$ is CULF monoidal with disjoint union $(+)$.*



The simplicial groupoid $T\mathbf{S}\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Grpd}$

Corollary

$\mathbb{Q}_{\pi_0 T\mathbf{S}}$ *has a bialgebra structure given the homotopy cardinality of the incidence bialgebra of* $T\mathbf{S}$*.*

Remark

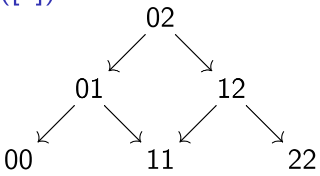
The isomorphism classes of **connected** elements

$$\begin{array}{ccc} & t_{01} & \\ \swarrow\!\!\swarrow & & \searrow\!\!\searrow \\ t_{00} & \twoheadrightarrow & 1 \end{array}$$

of $T_1\mathbf{S}$ form a basis of $\mathbb{Q}_{\pi_0 T_1\mathbf{S}}$.

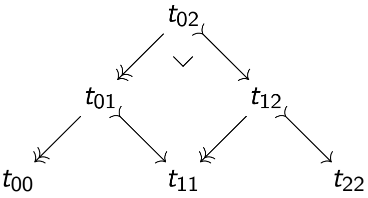
Formal construction of $T\mathbf{S}$

Twisted arrow category Tw([2])

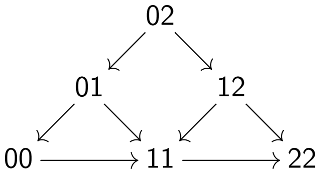


Quillen's Q-construction

$Q_n\mathcal{A} \subseteq \mathrm{Fun}^{\simeq}(\mathrm{Tw}([n]), \mathcal{A})$

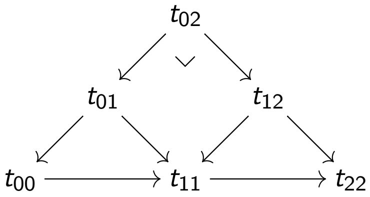


Extended twisted arrow category $\mathrm{Tw}^+([2])$



T-construction

$T_n\mathbf{S} \subseteq \mathrm{Fun}^{\simeq}(\mathrm{Tw}^+([n]), \mathbf{S})$



Theorem (C.)

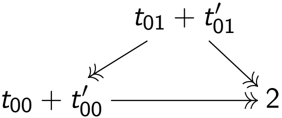
The homotopy cardinality of the incidence bialgebra of $T\mathbf{S}$ is isomorphic to the plethystic bialgebra.

The isomorphism $\mathbb{Q}_{\pi_0 T_1 \mathbf{S}} \simeq \mathcal{P}$

$$\begin{array}{ccc} & t_{01} & \\ \swarrow\!\!\!\!\swarrow & & \searrow\!\!\!\!\searrow \\ t_{00} & \twoheadrightarrow & 1 \end{array} \quad \mapsto A_\lambda,$$

where $\lambda = (\lambda_1, \lambda_2, \dots)$ represents the isomorphism class of

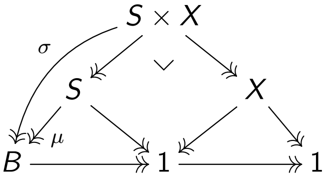
$$t_{01} \twoheadrightarrow t_{00}.$$



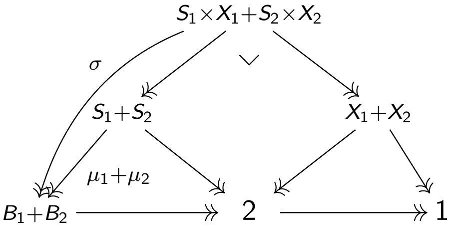
$\mapsto A_\lambda A_{\lambda'}$,

where λ and λ' represent the isomorphism classes of $t_{01} \twoheadrightarrow t_{00}$ and $t'_{01} \twoheadrightarrow t'_{00}$.

Verschiebung operator

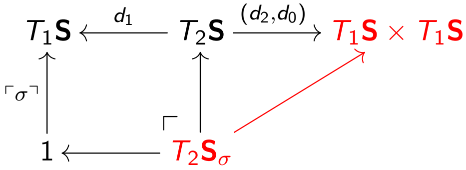


$$\sigma = V^{|X|}\mu$$



$$\sigma = V^{|X_1|}\mu_1 + V^{|X_2|}\mu_2$$

The comultiplication



$$\Delta(A_\sigma) = \Delta(|\ulcorner\sigma\urcorner|) = |\, T_2\mathbf{S}_\sigma \longrightarrow T_1\mathbf{S} \times T_1\mathbf{S}|$$

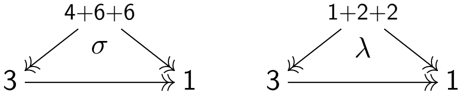
$$= \sum_{\lambda \in \pi_0 T_1\mathbf{S}} \sum_{\mu \in \pi_0 T_1\mathbf{S}} \frac{|\, T_2\mathbf{S}_{\sigma,\lambda,\mu}|}{|\operatorname{Aut}(\lambda)||\operatorname{Aut}(\mu)|} A_\mu \otimes A_\lambda$$

Hence we should see that

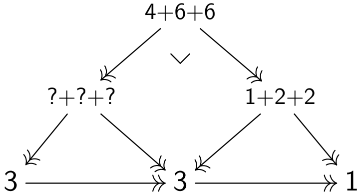
$$P_{\sigma,\lambda}(\{A_\mu\}) = \sum_{\mu \in \pi_0 T_1\mathbf{S}} \frac{|\, T_2\mathbf{S}_{\sigma,\lambda,\mu}|}{|\operatorname{Aut}(\lambda)||\operatorname{Aut}(\mu)|} A_\mu \otimes A_\lambda$$

Example

$$P_{(0,0,0,1,0,2),(1,2)} = \frac{6!^2 2!4!}{4!3!^2 2!^2 2!} A_{(0,0,0,1)} A^2_{(0,0,1)} + \frac{6!^2 2!4!}{6!3!2!2!^2 2!} 2A_{(0,0,0,0,0,1)} A_{(0,0,1)} A_{(0,1)}$$

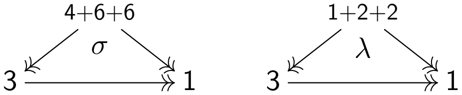


$$(0,0,0,1,0,2) = V^1\ ? + V^2\ ? + V^2\ ?$$

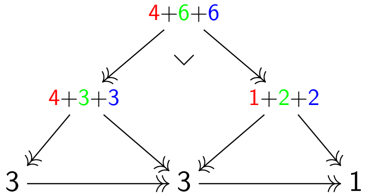


Example

$$P_{(0,0,0,1,0,2),(1,2)} = \frac{6!^2 2!4!}{4!3!^2 2!^2 2!} A_{(0,0,0,1)} A^2_{(0,0,1)} + \frac{6!^2 2!4!}{6!3!2!} 2A_{(0,0,0,0,0,1)} A_{(0,0,1)} A_{(0,1)}$$

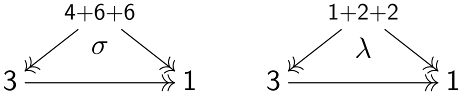


$$(0,0,0,1,0,2) = V^1(0,0,0,1) + V^2(0,0,1) + V^2(0,0,1)$$

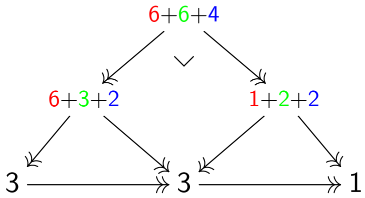


Example

$$P_{(0,0,0,1,0,2),(1,2)} = \frac{6!^2 2!4!}{4!3!^2 2!^2 2!} A_{(0,0,0,1)} A^2_{(0,0,1)} + \frac{6!^2 2!4!}{6!3!2!} 2A_{(0,0,0,0,0,1)} A_{(0,0,1)} A_{(0,1)}$$



$$(0,0,0,1,0,2) = V^1(0,0,0,0,0,1) + V^2(0,0,1) + V^2(0,1)$$



Thank you