A synthetic account of Huygens' Principle

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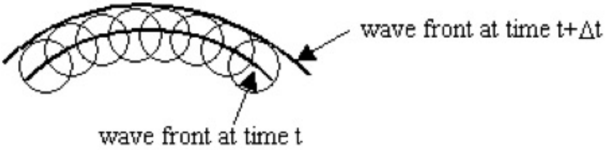
1. Huygens' Principle



Wave front at time $t + \Delta t$ = *envelope* of *wavelets* of radius Δt.
i.e. it is *touched* be each of the wavelets.

2. Primitive notions

Ambient space: M with

- a reflexive symmetric relation $\sim$ ("neighbour relation")
- a (pre-) metric dist on M

They define, respectively, the notions of

touching (and hence *envelope*)
sphere /circle

$$\sim$$

for x and y in M:

$$x \sim x$$

$$x \sim y \ \Rightarrow y \sim x.$$

The "first neighbourhood of the diagonal" $M_{(1)} \subseteq M \times M$.

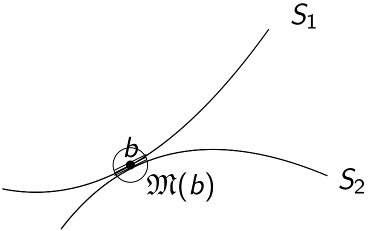
$\sim$ is not transitive, - unlike in NSA.

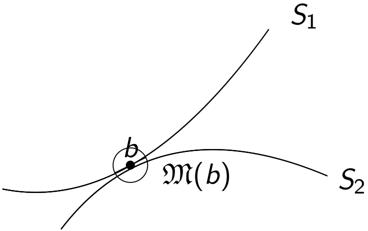
2.1 Touching

- Monad around $b \in M$
$\mathfrak{M}(b) := \{b' \in M \mid b' \sim b\}$

Let $b \in S_1 \cap S_2$.

- S_1 and S_2 *touch* at b: $\mathfrak{M}(b) \cap S_1 = \mathfrak{M}(b) \cap S_2$





Equivalently, $b \in S_1 \cap S_2$, and

$$\forall b' \sim b : b' \in S_1 \Leftrightarrow b' \in S_2.$$

Touching *set* of S_1 and S_2 = set of points where S_1 and S_2 touch.

In general a *proper* subset of $S_1 \cap S_2$.

3. Neighbours and touching in SDG

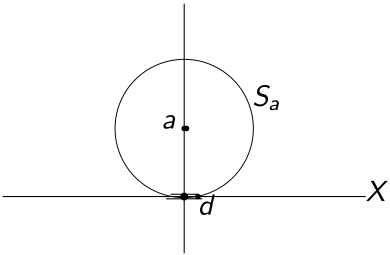
(for motivation only):
R: basic number line; a commutative ring. R^2: the coordinate plane, etc.
In R, we have neighbour relation $x \sim y \Leftrightarrow (y - x)^2 = 0$.

For any space M, we may define $\sim$ on M by
$\underline{x} \sim \underline{y}$: for all $\phi : M \to R$, $\phi(\underline{x}) \sim \phi(\underline{y})$.

Then any map $M \to N$ preserves $\sim$ ("automatic continuity").

$\sim$ is reflexive and symmetric, *but not transitive*.

Protagoras' Picture



The point $(d, 0)$ on the x-axis X has distance a to $(0, a)$ iff
$d^2 = 0$, (by Pythagoras' Theorem)
i.e. iff $d \sim 0$.
$S_a \cap X$ is the little "line element", containing e.g. $(d, 0)$.
But $(0, 0)$ is the only *touching* point of S_a and X.

4. Pre-metric dist

For x and y in M:
$\text{dist}(x, y) \in R_{>0}$

only defined for x *distinct from* y
(if $x \sim y$, x and y are not distinct !)

Symmetric: $\text{dist}(x, y) = \text{dist}(y, x)$.

Assumptions for $R_{>0}$:
An (additively written) cancellative semigroup
Define $r < t$ to mean:
$\exists s : r + s = t$ (equivalently $\exists! s : r + s = t$)
This unique s is the *difference* $t - r$.

Require *dichotomy* for the natural strict order $>$ on $R_{>0}$:
if r and s are distinct, then
either $r < s$ or $s < r$.

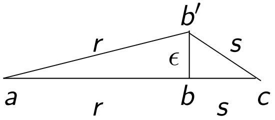
No triangle inequality is assumed.
But we may for *some* triples a, b, c in M have triangle *equality*:

$$\mathrm{dist}(a, b) + \mathrm{dist}(b, c) = \mathrm{dist}(a, c)$$

(a *weak* collinearity condition for a, b, c).

Busemann 1943: On Spaces in which Two Points determine a Geodesic.
Busemann 1969: Synthetische Differentialgeometrie.

“Plucked string”-picture



Spheres

Let M be a space equipped with a (pre-) metric.
Let $a \in M$ and $r \in R_{>0}$. Define

$$S(a, r) := \{b \in M \mid \text{dist}(a, b) = r\},$$

the *sphere* with center a and radius r.

Nonconcentric spheres: their centers are *distinct*.

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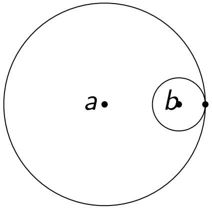
5. The Axioms

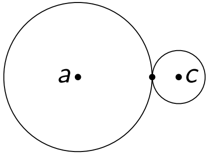
- Axiom 1: If two spheres touch, there is a *unique* touching point.

- Axiom 2:

Two spheres touch iff *either*
the distance between their centers equals the *difference* between their radii ("concave touching")



or the distance between their centers equals the *sum* of their radii ("convex touching")

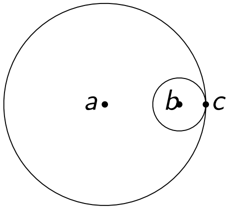


- Axiom 3 ("Dimension Axiom")

Given two spheres S_1 and S_2, and $b \in S_1 \cap S_2$. Then

$$\mathfrak{M}(b) \cap S_1 \subseteq \mathfrak{M}(b) \cap S_2 \quad \text{implies} \quad \mathfrak{M}(b) \cap S_1 = \mathfrak{M}(b) \cap S_2.$$

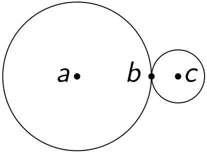
Difference of radii:



Denote the touching point c; then :

$$\text{dist}(a, b) = \text{dist}(a, c) - \text{dist}(b, c).$$

Sum of radii:



Denote the touching point b; then :

$$\mathrm{dist}(a, c) = \mathrm{dist}(a, b) + \mathrm{dist}(b, c).$$

So

$$\mathrm{dist}(a, b) = \mathrm{dist}(a, c) - \mathrm{dist}(b, c)$$

$$\mathrm{dist}(a, c) = \mathrm{dist}(a, b) + \mathrm{dist}(b, c)$$

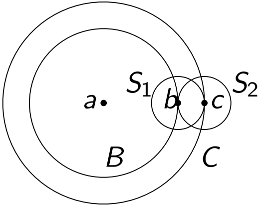
are thus *necessary* conditions for c and b being the respective touching points. These two “arithmetical” necessary conditions are trivially equivalent.

6. Reciprocity Lemma

Let a, b, c satisfy the triangle equality

$$dist(a, c) = dist(a, b) + dist(b, c).$$

Then b is the touching point of B and S_2 iff c is the touching point of C and S_1



We say that a, b, c are *strongly collinear* it they are weakly collinear (triangle equality holds): $\mathrm{dist}(a,b) + \mathrm{dist}(b,c) = \mathrm{dist}(a,c)$ (write $r := \mathrm{dist}(a,b)$, $s := \mathrm{dist}(b,c)$)

and

b is the touching point of $S(a,r)$ and $S(c,s)$ (convex)

equivalently, by Reciprocity Lemma,

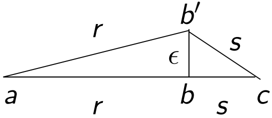
c is the touching point of $S(a,r+s)$ and $S(b,s)$ (concave)

spelled out in 1st order terms:

$$\forall b' \sim b : \mathrm{dist}(a,b') = r \Leftrightarrow \mathrm{dist}(b',c) = s$$

respectively

$$\forall c' \sim c : \mathrm{dist}(a,c') = r + s \Leftrightarrow \mathrm{dist}(b,c') = s$$



For $\epsilon^2 = 0$, a, b', c are weakly collinear,

so $S(a, r)$ and $S(c, s)$ do touch,
but not in b'; they touch in b.
So a, b, c are strongly collinear

Recall

$$\forall b' \sim b : \mathrm{dist}(a, b') = r \Leftrightarrow \mathrm{dist}(b', c) = s$$

as condition for $S(a, r)$ touching $S(c, s)$ in b.

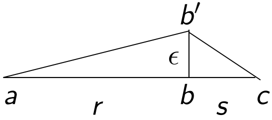
By Dimension Axiom 3, $\Leftrightarrow$ may be replaced by $\Rightarrow$, or by $\Leftarrow$. Then we get some equivalent formulations. E.g.

$$\forall b' \sim b : \mathrm{dist}(a, b') = r \Rightarrow \mathrm{dist}(b', c) = s.$$

or in terms used in calculus:

- b is a critical point of the function $\mathrm{dist}(x, c)$ under the constraint $\mathrm{dist}(a, x) = r$; with critical value s

By a *critical point* of a function $\phi : M \to R_{>0}$, we mean a point $x \in M$ so that ϕ is constant on $\mathfrak{M}(x)$.
If $B \subseteq M$, *a critical point of* ϕ *under the constraint* $x \in B$, is a point $x \in B$ so that ϕ is constant on $\mathfrak{M}(x) \cap B$.



If $\epsilon^2 = 0$, $\text{dist}(a, b') = r$ and $\text{dist}(b', c) = s$, so both "paths" from a to c have length $r + s$.
"Shortest path" is not enough to characterize (strong) collinearity!

• b is *the* critical point of the function $\text{dist}(x, c)$ under the constraint $\text{dist}(a, x) = r$; with critical value s

7. Contact elements

A *contact element* P *at* $b \in P$ *is a subset which may be written*

$$\mathfrak{M}(b) \cap S$$

for some sphere S *containing* b.

The sphere S is said to *represent* the contact element.

If two spheres S_1 and S_2 touch each other at b

$$\mathfrak{M}(b) \cap S_1 = \mathfrak{M}(b) \cap S_2.$$

So if S_1 represents (P, b), then so does S_2.

Let $P = (P, b)$ be a contact element. Let S be a sphere. If $P \subseteq S$, then S represents P.
For, let S_1 be a sphere representing P. Then $\mathfrak{M}(b) \cap S_1 \subseteq \mathfrak{M}(b) \cap S$. By Axiom 3, have equality.

In the applications, when M is 2-dimensional, the contact elements may be called: *line* elements, and if M is 3-dimensional, they may be called *plane* elements.

A contact element in an n-dimensional M is of dimension $n - 1$. The set of contact elements in M make up "the projectivized cotangent bundle of M".

$$\mathfrak{M}(b) \cap S$$

Sophus Lie: "It is often *practically* convenient to think of a line element as an infinitely small piece of a curve."

Zur Theorie partieller Differentialgleichungen, 1872
Berührungstransformationen, 1896

Berührung = touching = contact

Perpendicularity, and the normal

Given $P = (P, b)$.
Let $x \in M$ be distinct from b
(equivalently, distinct from all points of P). We define

$$x \perp P :\Leftrightarrow [\mathrm{dist}(x, b') = \mathrm{dist}(x, b) \text{ for all } b' \in P].$$

The set of points x with $x \perp P$ make up the *normal* $P^\perp$ to P.

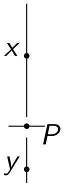
$P = (P, b)$. Recall

$$x \perp P :\Leftrightarrow [\mathrm{dist}(x, b') = \mathrm{dist}(x, b) \text{ for all } b' \in P].$$

Expressed in terms of spheres: $P \subseteq S(x, s)$, where $s = \mathrm{dist}(x, b)$. Equivalently: $S(x, s)$ represents P.

If x_1 and x_2 are $\perp P$, then they are strongly collinear with b. For $P \subseteq S(x_1, s_2)$ and $P \subseteq S(x_2, s_2)$. So both these spheres represent P.
If two spheres S_1 and S_2 represent (P, b), then they touch each other at b. Assume e.g. convex touching. Then a, b, c are strongly collinear, where a is the center of S_1 and c is the center of S_2.

(Contrast with the discrete case where *all* points (distinct from b) are $\perp \{b\}$.)



For x and y on the normal, we say that they are on the *opposite side* of P if

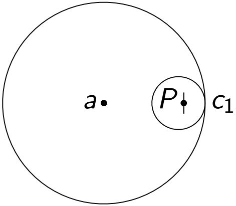
$$\text{dist}(x, y) > \text{dist}(x, b) \text{ and } \text{dist}(x, y) > \text{dist}(y, b),$$

otherwise we say that that they are on the *same* side.
The normal $P^{\perp}$ falls in two subsets; selecting one of these as the "positive normal" provides P with a (transversal) *orientation*
A sphere representing a transversally oriented P *represents it from the inside* if its center belongs to the negative normal.

Crucial construction: $P \vdash s$

"The" point obtained by going s units out along the positive normal of $P = (P, b)$.
Two constructions:

$P = (P, b)$



Pick $S = S(a, r)$ representing P from the inside. (Only $\mathfrak{M}(b) \cap S = P$ is visible!)
Let c_1 be the touching point of $S(a, r + s)$ and $S(b, s)$.
By Reciprocity, b is the touching point of $S(a, r)$ and $S(c_1, s)$,
so $P = \mathfrak{M}(b) \cap S(a, r) \subseteq S(c_1, s)$
so $c_1 \perp P$, and $\mathrm{dist}(b, c_1) = s$.
So there *exist* points on the positive normal of P with prescribed distance s.

Independent of choice of a sphere $S(a, r)$ representing P?

Assume c_2 has $\text{dist}(b, c_2) = s$ and $c_2 \perp P$. (This condition is independent of a and r!)

So $\text{dist}(b', c_2) = s$ for all $b' \in P$. Equivalently, $P \subseteq S(c_2, s)$.

Pick a sphere $S(a, r)$ so that $P = \mathfrak{M}(b) \cap S(a, r)$, i.e.
$\mathfrak{M}(b) \cap S(a, r) = P \subseteq S(c_2, s)$

so $\mathfrak{M}(b) \cap S(a, r) \subseteq S(c_2, s)$, and by Dimension Axiom 3,
$\mathfrak{M}(b) \cap S(a, r) = \mathfrak{M}(b) \cap S(c_2, s)$
b is the touching point of $S(a, r)$ and $S(c_2, s)$.

Hence by Reciprocity Lemma, c_2 is the touching point of $S(a, r + s)$ and $S(b, s)$.
So $c_1 = c_2$.

$$c_2 \in \bigcap_{b' \in P} S(b', s)$$

Discriminant-type construction of the characteristic point of the family of spheres $\{S(b', s) \mid b' \in P\}$.

8. Hypersurfaces

A *hypersurface* B: a subset $B \subseteq M$ such that for each $b \in B$, $\mathfrak{M}(b) \cap B$ is a contact element:

$$B(b) := \mathfrak{M}(b) \cap B.$$

B is *oriented* if each $B(b)$ is oriented.

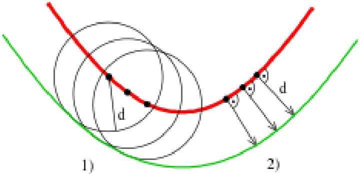
$$B \vdash s := \{B(b) \vdash s \mid b \in B\}.$$

Have map $B \to B \vdash s$: $b \mapsto B(b) \vdash s (= C)$.
For small enough s, this map is a bijection.
The inverse is the *foot* map. Require that it extends to a neigbourhood of C.

Then can prove

$$\mathfrak{M}(c) \cap S(b, s) = \mathfrak{M}(c) \cap C.$$

This proves that C is a hypersurface: witnessed by the wavelets $S(b, s)$.
It also proves: C is en envelope of the wavelets.



Let $c \in C$ have foot b on B
To prove $\mathfrak{M}(c) \cap S(b, s) \subseteq C$:

Let $x \in \mathfrak{M}(c) \cap S(b, s)$.
Let b' be the foot of x on B.
Since $x \sim c$, we have $b' \sim b$.
Since b' is foot of x and $b' \sim b$, we have
$s = \mathrm{dist}(x, b) = \mathrm{dist}(x, b')$.
So $x = B(b') \vdash s$, hence b' witnesses that $x \in C$.

Needs attention:

Coexistence with *Riemannian* metric?

Synthetically, a Riemannian metric on M is a function $g : M_{(2)} \to R$, vanishing in $M_{(1)} =$ the set of pairs of neighbour points, where $M_{(2)}$ is the set of points which are second-order neighbours, e.g. on R: pairs (x, y) with $(y - x)^3 = 0$. Interpret $g(x, y)$ as "square-of-distance".

Purely combinatorial models ? e.g. with $R_{>0} :=$ positive integers ?