Algebraic Structure From Non-Algebraic Proofs

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Definition (Grandis, Tholen)

An *algebraic weak factorisation system* (awfs) (a.k.a natural weak factorisation system) on a category $\mathbb{C}$ consists of a comonad $\mathsf{L}\colon \mathbb{C}^{\mathbf{2}} \to \mathbb{C}^{\mathbf{2}}$ and monad $\mathsf{R}\colon \mathbb{C}^{\mathbf{2}} \to \mathbb{C}^{\mathbf{2}}$ where the underlying copointed and pointed endofunctors arise from a functorial factorisation, satisfying a "distributive law."

We can used awfs's to get a structured notion of (trivial) cofibrations and fibrations. What about weak equivalences?

Definition

Suppose we are given a morphism of awfs's $\xi \colon (\mathsf{C}^t, \mathsf{F}) \to (\mathsf{C}, \mathsf{F}^t)$.

We say a *structured weak equivalence* is a map $f \colon X \to Y$ together with an F^t-algebra structure on $\mathsf{F}f$.

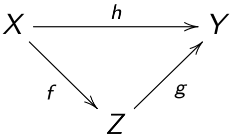
A *morphism of structured weak equivalences* is a commutative square

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \longrightarrow & Y' \end{array}$$

such that the induced map $\mathsf{F}f \to \mathsf{F}f'$ is a morphism of F^t-algebras.

Write the resulting category as W-**Map**

Consider diagrams of the following form:



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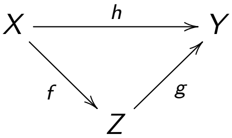

A *functorial 3-for-2 operator* takes such a diagram together with weak equivalence structures on two of the maps, and returns a weak equivalence structure on the third map, preserving morphisms of structured weak equivalences.

Definition

An *algebraic model structure with structured weak equivalences* consists of

1. A morphism of algebraic weak factorisation systems $\xi\colon (\mathsf{C}^t, \mathsf{F}) \to (\mathsf{C}, \mathsf{F}^t)$.
2. A functorial 3-for-2 operator.

Functors of the form below, that commute with forgetful functors.

1. C-**coalg** $\times_{\mathbb{C}^2}$ W-**Map** $\to$ C^t-**coalg**
2. C^t-**coalg** $\to$ W-**Map**
3. F-**Alg** $\times_{\mathbb{C}^2}$ W-**Map** $\to$ F^t-**Alg**
4. F^t-**Alg** $\to$ W-**Map**

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Theorem (S)

Suppose that $\mathbb{C}$ is a category with an ams with structured weak equivalences and a "stable functorial choice of path objects." Then $\mathbb{C}$ also has a "stable functorial choice of very good path objects" (which can be used for identity types, giving explicit definitions for J-terms).

- This is based on the notion of algebraic model structure due to Riehl.
- The original motivation for this definition was the construction of identity types in cubical sets.
- Unpublished results by Sattler suggest a wide range of categories (including CCHM cubical sets) can be made in algebraic model structures with structured weak equivalences.

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Theorem (Sattler)

Suppose $(\mathbb{C}, \otimes)$ *is a finitely complete and finitely cocomplete symmetric affine monoidal closed category and we are given*

1. *an interval object* $\delta_0, \delta_1 : 1 \to \mathbb{I}$
2. *wfs's* $(\mathcal{C}, \mathcal{F}^t)$ *and* $(\mathcal{C}^t, \mathcal{F})$
3. $\mathcal{C}^t \subseteq \mathcal{C}$
4. $\mathcal{C}$ *is closed under pullbacks*
5. f *is a fibration if and only if* $\delta_0 \hat{\oslash} f$ *and* $\delta_1 \hat{\oslash} f$ *are trivial fibrations.*
6. $[\delta_0, \delta_1] \hat{\oslash} -$ *preserves trivial fibrations.*
7. *Trivial cofibrations are stable under pullback along fibrations*
8. *(Trivial) Fibrations extend along trivial cofibrations*

Define $\mathcal{W}$ *to be maps of the form* $f \circ m$ *where* $m \in \mathcal{C}^t$ *and* $f \in \mathcal{F}^t$*. Then* $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ *is a model structure.*

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Usually, the most natural way to prove 5 is to show $\mathcal{C}$ is generated by a set I and $(\mathcal{C}^t, \mathcal{F})$ is cofibrantly generated by maps $\delta_i \hat{\otimes} m$ where $m \in I$ and $i = 0, 1$.

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We will make this idea precise using Grothendieck fibrations.

1. Making it easy to formalise (without missing out too many details, or leaving them as exercises for the reader).
2. (Hopefully) The same ideas apply to variants based on realizabilty, allowing us to extract computational information telling us how to compute the operators.

Definition

Let $\mathbb{C}$ be a category. We define the Grothendieck fibration of category indexed families $p\colon \mathrm{Fam}(\mathbb{C}) \to \mathbf{CAT}$ as follows. An object of $\mathrm{Fam}(\mathbb{C})$ consists of $\mathbb{A} \in \mathbf{CAT}$ together with a functor $\mathbb{A} \to \mathbb{C}$. p is defined to be the projection functor.

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For the proof to work, we need $\mathbf{CAT}$ to contain categories the same cardinality as $\mathbb{C}$ (so in particular $\mathbb{C} \in \mathbf{CAT}$).

By a well known result due to Freyd, $\mathbb{C}$ cannot have colimits of shape $\mathbb{A}$ for all $\mathbb{A} \in \mathbf{CAT}$ unless it is a poset. Hence in this case p is not a bifibration.

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3. $(\mathsf{C}, \mathsf{F}^t)$ *is stable under pullback*
4. $(\mathsf{C}, \mathsf{F}^t)$ *is cofibrantly generated by a functor* $M \colon J \to \mathbb{C}^{\mathbf{2}}$
5. $(\mathsf{C}^t, \mathsf{F})$ *is cofibrantly generated by* $\delta_0 \hat{\otimes} M + \delta_1 \hat{\otimes} M$
6. *There is an endomorphism* $E \colon J \to J$ *such that* $M \circ E \cong [\delta_0, \delta_1] \hat{\otimes} M$
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$(\mathsf{C}^t, \mathsf{F})$ and $(\mathsf{C}, \mathsf{F}^t)$ uniquely extend to fibred awfs's over $\mathrm{Fam}(\mathbb{C}) \to \mathbf{CAT}$. We just define them pointwise.

(Note that each fibre category $\mathrm{Fam}(\mathbb{C})_{\mathbb{A}}$ is the functor category $[\mathbb{A}, \mathbb{C}]$ and the restrictions of $(\mathsf{C}, \mathsf{F}^t)$ and $(\mathsf{C}^t, \mathsf{F})$ are the pointwise awfs's.)

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We then translate all of definitions to properties of the fibration.

Proposition

Suppose G is a vertical map over the category $\mathbb{A}$. Then

1. *We can view G as a functor $\mathbb{A} \to \mathbb{C}^{\mathbf{2}}$*
2. F*-algebra structures on G correspond to a choice of* F*-algebra structure on $G(A)$ for each $A \in \mathbb{A}$ such that $G(\sigma)$ is a morphism of* F*-algebras for each morphism σ of $\mathbb{A}$.*
3. *Similarly for* F^t*-algebra,* C*-coalgebra,* C^t*-coalgebra, and weak equivalence structures.*

For the cofibrantly generated parts we use the following definition.

Definition

Suppose $p \colon \mathbb{E} \to \mathbb{B}$ is a Grothendieck fibration, and (L, R) is a fibred awfs over p.
Suppose M is a vertical map in the fibre of J and G is a vertical map in the fibre of I.

A *family of lifting problems* from M to G is a map $\sigma \colon K \to J$ in $\mathbb{B}$ together with a lifting problem from $\sigma^*(M)$ to G.

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A *family of lifting problems* from M to G is a map $\sigma \colon K \to J$ in $\mathbb{B}$ together with a lifting problem from $\sigma^*(M)$ to G.

We say (L, R) is *cofibrantly generated* by a vertical map M if for all vertical maps G, R-algebra structures on G correspond naturally to coherent choices of fillers for all families of lifting problems from M to G.

We obtain the following structure over $p\colon \mathrm{Fam}(\mathbb{C}) \to \mathbf{CAT}$.

1. an interval object $\delta_0, \delta_1 \colon 1 \to \mathbb{I}$ over 1
2. Fibred awfs's (C^t, F) and (C, F^t)
3. (C, F^t) is stable under pullback
4. (C, F^t) is cofibrantly generated by a vertical map M
5. (C^t, F) is cofibrantly generated by $\delta_0 \hat{\otimes} M + \delta_1 \hat{\otimes} M$
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For every $\mathbb{A}$, the fibre category $\mathrm{Fam}(\mathbb{C})_{\mathbb{A}}$ satisfies the conditions to apply Sattler's non-algebraic result. We deduce.

Lemma

For every category $\mathbb{A} \in \mathbf{CAT}$, $\mathrm{Fam}(\mathbb{C})_{\mathbb{A}}$ *is a model structure.*

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We define $\mathbb{A}$ to be the category with objects pairs of structured weak equivalences $X \xrightarrow{g} Y \xrightarrow{h} Z$ in $\mathbb{C}$.
A morphism of $\mathbb{A}$ is a diagram of the form below, where both squares preserve the weak equivalence structures.

$$\begin{array}{ccccc} X & \xrightarrow{g} & Y & \xrightarrow{h} & Z \\ \downarrow & & \downarrow & & \downarrow \\ X' & \xrightarrow[g']{} & Y' & \xrightarrow[h']{} & Z' \end{array}$$

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This defines two vertical maps G and H in $\mathrm{Fam}(\mathbb{C})_{\mathbb{A}}$, and moreover they are weak equivalences in $\mathrm{Fam}(\mathbb{C})_{\mathbb{A}}$. Since $\mathrm{Fam}(\mathbb{C})_{\mathbb{A}}$ is a model structure $H \circ G$ must also be a weak equivalence in $\mathrm{Fam}(\mathbb{C})_{\mathbb{A}}$. We deduce that we can assign $h \circ g$ the structure of a weak equivalence for each object (g, h) in $\mathbb{A}$, and moreover this assignment is functorial.

- This result can be used to show BCH cubical sets form an ams with structured weak equivalences.
- It may lead to a more efficient proof of Sattler's result that CCHM cubical sets (and many other categories) form an ams with structured weak equivalences.
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Thank you for your attention!