A new characterisation of higher central extensions in semi-abelian categories

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Outline

Introduction

The concept of higher centrality is useful and unavoidable in the recent approach to homology and cohomology of non-abelian structures based on categorical Galois theory. In our work, higher central extensions are the covering morphisms with respect to certain Galois structures induced by a refletion

$$\mathbb{X} \underset{\supset}{\overset{\mathsf{Ab}}{\underset{\perp}{\rightleftarrows}}} \mathsf{Ab}(\mathbb{X})$$

and can also be defined more generally, for any semi-abelian category $\mathbb{X}$ and any Birkhoff subcategory $\mathscr{B}$ of $\mathbb{X}$. The descriptions of higher central extensions in terms of algebraic conditions using "generalised commutators" is in general a non-trivial problem.

Today, I am going to:

- give a new characterisation of higher central extensions in terms of higher-order Higgins commutators in semi-abelian categories which do not satisfy the *Smith is Huq* condition.
- give some perspectives for future work.

Semi-abelian categories

Throughout this presentation, $\mathbb{X}$ is a semi-abelian category.

Definition [G. Janelidze, L. Márki, and W. Tholen]

A category $\mathbb{X}$ is semi-abelian when it

1. is pointed;
2. has binary coproducts;
3. is Barr-exact;
4. is Bourn-protomodular: the Split Short Five Lemma holds.

Examples: Grp, Lie_K, Alg_K, XMod, varieties of Ω-groups, Loops, Near-Rings.

Definition [G. Janelidze, G.M. Kelly]

A subcategory $\mathscr{B}$ of $\mathbb{X}$ is a Birkhoff subcategory when it is closed under subobjects and regular quotients.

Examples:
- Any subvariety $\mathscr{B}$ of a variety of universal algebras $\mathscr{V}$.
- The subcategory $\mathrm{Ab}(\mathbb{X})$ of abelian objects in $\mathbb{X}$.

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Higher extensions

- An n-**fold arrow** in $\mathbb{X}$ is a functor $F : (2^n)^{op} \longrightarrow \mathbb{X}$.

$$\mathrm{Arr}^n(\mathbb{X}) = \mathrm{Fun}((2^n)^{op}, \mathbb{X})$$

- An n-fold arrow F is an n-**fold extension** when for all $\varnothing \neq I \subseteq n$ the arrow
$F_I \twoheadrightarrow \lim_{J \subsetneq I} F_J$ is a regular epimorphism. $\mathrm{Ext}^n(\mathbb{X})$ is the category of n-fold extensions
- The adjunction

$$\mathbb{X} \underset{\supset}{\overset{\mathrm{ab}}{\underset{\perp}{\rightleftarrows}}} \mathrm{Ab}(\mathbb{X})$$

induces a Galois structure

$$\Gamma_0 = (\mathbb{X}, \mathrm{Ab}(\mathbb{X}), \mathrm{ab}, \subset, \mathscr{E}, \mathscr{F})$$

in the sense of **G. Janelidze.**
- A 1-fold extension $f : B \twoheadrightarrow A \in \mathscr{E}$ is **central** w.r.t Γ_0 if and only if the square

is a pullback.

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- A 1-fold extension $f : B \twoheadrightarrow A \in \mathscr{E}$ is **central** w.r.t Γ_0 if and only if the square

$$\begin{array}{ccc} \mathrm{Eq}(f) & \xrightarrow{\pi_1} & B \\ {\scriptstyle \eta^0_{\mathrm{Eq}(f)}} \downarrow & \lrcorner & \downarrow {\scriptstyle \eta^0_B} \\ \mathsf{ab}_0(\mathrm{Eq}(f)) & \xrightarrow[\mathsf{ab}(\pi_1)]{} & \mathsf{ab}_0(B) \end{array}$$

is a pullback.

Higher central extensions

- The category $\mathsf{CExt}(\mathbb{X})$ of 1-fold central extensions in $\mathbb{X}$ is a strongly $\mathscr{E}^1$-Birkhoff subcategory of $\mathsf{Ext}(\mathbb{X})$
- Inductively, for any $n \geqslant 1$, this gives an adjunction

$$\mathsf{Ext}^n(\mathbb{X}) \underset{\supset}{\overset{\mathsf{ab}_n}{\rightleftarrows}} \mathsf{CExt}^n(\mathbb{X}) \quad (\bot)$$

which induce a Galois structure

$$\Gamma_n = (\mathsf{Ext}^n(\mathbb{X}), \mathsf{CExt}^n(\mathbb{X}), \mathsf{ab}_n, \subset, \mathscr{E}^n, \mathscr{F}^n)$$

- The category $\mathsf{CExt}^n(\mathbb{X})$ of n-fold central extensions in $\mathbb{X}$ w.r.t Γ_{n-1} is a strongly $\mathscr{E}^n$-Birkhoff subcategory of $\mathsf{Ext}^n(\mathbb{X})$
- An n-fold extension $f : B \to A$ is central w.r.t Γ_{n-1} if and only if the square

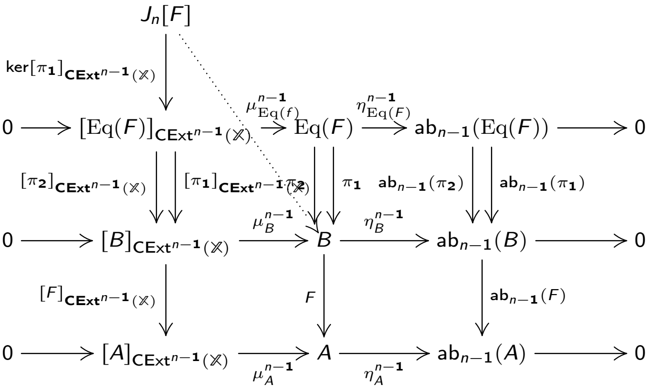
$$\begin{array}{ccc} \mathrm{Eq}(f) & \xrightarrow{\pi_1} & B \\ {\scriptstyle \eta^{n-1}_{\mathrm{Eq}(f)}}\downarrow & & \downarrow{\scriptstyle \eta^{n-1}_B} \\ \mathsf{ab}_{n-1}(\mathrm{Eq}(f)) & \xrightarrow[\mathsf{ab}_{n-1}(\pi_1)]{} & \mathsf{ab}_{n-1}(B) \end{array}$$

is a pullback.

The reflection

The reflection $\mathsf{ab}_n : \mathsf{Ext}^n(\mathbb{X}) \to \mathsf{CExt}^n(\mathbb{X})$ is built as follows:

[T. Everaert, M. Gran, and T. Van der Linden, 2008]



The object $L_n[F]$

This yields a morphism of short exact sequences in $\mathrm{Arr}^{n-1}(\mathbb{X})$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_n[F] & \longrightarrow & B & \xrightarrow{\rho_F^n} & \mathrm{ab}_n[F] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle J_nF} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \mathrm{ab}_nF} & & \\ & & 0 & \longrightarrow & A & \cdots\cdots & A & \longrightarrow & 0 \end{array}$$

- $L_n[F]$ is the inital object of the n-fold extension J_nF denoted by

$$L_n[F] = (J_nF)_n$$

- J_nF is zero everywhere, except on its initial object $L_n[F]$.

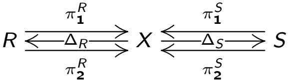
Remark [T. Everaert, M. Gran, and T. Van der Linden, 2008]

An n-fold extension F is central w.r.t Γ_{n-1} if and only if $L_n[F] = 0$

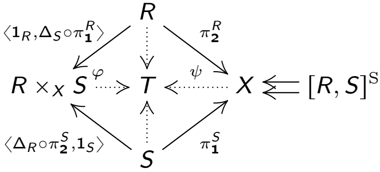
- What is $L_n[F]$? Our goal is to give an explicite description of this object in terms of "generalised commutators".

The Smith is Huq condition

For equivalence relations R, S on X



The **Smith-Pedicchio commutator** $[R, S]^{\mathrm{S}}$, is the kernel pair of ψ



For subobjects K, L of X, the **Huq-Bourn commutator** $[K, L]_Q$ is the kernel of the morphism q,

- R and S **Smith commute** iff

$$[R, S]^{\mathrm{S}} = \Delta_X$$

- K and L **Huq commute** iff $[K, L]_Q = 0$

The Smith is Huq condition

A semi-abelian category $\mathbb{X}$ satisfies the **Smith is Huq condition (SH)**, when two equivalence relations on the same object Smith-commute if and only if their normalisations Huq commute.

The Smith is Huq condition

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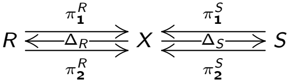

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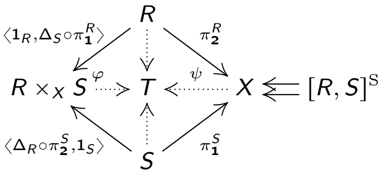

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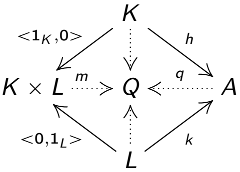

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The Smith is Huq condition

A semi-abelian category $\mathbb{X}$ satisfies the **Smith is Huq condition (SH)**, when two equivalence relations on the same object Smith-commute if and only if their normalisations Huq commute.

- When the condition (SH) holds, the object $L_n[F]$ has a characterisation in terms of binary Higgins or binary Huq commutators.

Examples of categories with (SH)

- Grp;
- Lie_K;
- Action accessible categories;
- Categories of interest in the sense of Orzech.

The known results

Definition [D.Rodelo and T. Van der Linden 2012]

A semi-abelian category $\mathbb{X}$ satisfies the **Commutators Condition** (CC) when: for all $n \geqslant 1$, an n-fold extension F is central if and only if

$$\bigvee_{I \subseteq n} [\bigwedge_{i \in I} K_i, \bigwedge_{i \in n \setminus I} K_i]_H = 0$$

Remark [D.Rodelo and T. Van der Linden 2012]

For semi-abelian categories : $(SH) \Rightarrow (CC)$

Theorem [D.Rodelo and T. Van der Linden 2012]

In any semi-abelian category $\mathbb{X}$ which satisfy the condition (SH), an n-fold extension F is central in the Galois theory sense if and only if

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For any n-fold extension F in a semi-abelian category which satisfy the condition (SH), we have

$$L_n[F] = \bigvee_{I \subseteq n} [\bigwedge_{i \in I} K_i, \bigwedge_{i \in n \setminus I} K_i]_H$$

- Our aim is to give a characterisation of $L_n[F]$ when the condition (SH) does not holds.

Examples of categories without (SH)

- Loops;
- Digroups;
- Near-Rings.

- To acheive our goal, we will need the concept of higher-order Higgins commutator.

The higher-order Higgins commutator

- Given objects $X_1, \cdots, X_n$, $n \geqslant 2$, in any finitely cocomplete homological category, their **co-smash product [A.Carboni, G. Janelidze]** $X_1 \diamond \cdots \diamond X_n$ is the kernel

$$X_1 \diamond \cdots \diamond X_n \xrightarrow{\tau_{X_1,\cdots,X_n}} \coprod_{j=1}^{n} X_j \xrightarrow{r_{X_1,\cdots,X_n}} \prod_{j=1}^{n} \Big(\coprod_{l=1, l\neq j}^{n} X_l \Big)$$

where $r_{X_1,\cdots,X_n}$ is the morphism determined by

$$\pi_{\coprod_{l=1,l\neq j}^{n} X_l} \circ r_{X_1,\cdots,X_n} = \begin{cases} \iota_{X_l} \text{ if } & l \neq j \\ 0 \text{ if } & l = j \end{cases}$$

- **For example:** when $n = 3$ and X, Y, Z are objects of $\mathbb{X}$, the co-smash product $X \diamond Y \diamond Z$ is defined as the kernel

$$X \diamond Y \diamond Z \xrightarrow{\tau_{X,Y,Z}} X + Y + Z \xrightarrow{\left\langle \begin{smallmatrix} \iota_X & \iota_X & 0 \\ \iota_Y & 0 & \iota_Y \\ 0 & \iota_Z & \iota_Z \end{smallmatrix} \right\rangle} (X+Y) \times (X+Z) \times (Y+Z)$$

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The higher-order Higgins commutator

Definition [M.Hartl]

Let X be an object of a finitely cocomplete homological category $\mathbb{X}$, and X_i be subobjects of X with their associated morphisms $x_i : X_i \to X$ for $1 \leqslant i \leqslant n$. Their n-**fold Higgins commutator** is the sub-object $[X_1, \cdots, X_n]_H$ given by:

$$\begin{array}{ccccc}
X_1 \diamond \cdots \diamond X_n & \xrightarrow{\tau_{X_1,\cdots,X_n}} & \coprod_{j=1}^{n} X_j & \xrightarrow{r_{X_1,\cdots,X_n}} & \prod_{j=1}^{n}\Big(\coprod_{l=1, l\neq j}^{n} X_l\Big) \\
\Big\downarrow & & \Big\downarrow \left\langle \begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix} \right\rangle & & \\
[X_1, \cdots, X_n]_H & \longrightarrow & X & &
\end{array}$$

- When $n = 2$, it coincides with the binary Higgins commutator introduced in any finitely cocomplete ideal determined category by G. Metere and S. Mantovani.
- When $n = 3$ and $\mathbb{X}$ is any algebraically coherent semi-abelian category, given normal subobjects K, L, M of an object G, their ternary Higgins commutator is given by :

$$[K, L, M]_H = [[K, L]_H, M]_H \vee [[M, K]_H, L]_H \vee [[L, M]_H, K]_H$$

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Motivation for our work

When we drop the Smith is Huq condition, we obtain the following characterisation of the Smith centrality of equivalence relations:

Proposition [M. Hartl and T. Van der Linden, 2013]

In a finitely cocomplete homological category, consider effective equivalence relations R and S on X with normalisations $K, L \lhd X$, respectively. Then

$$[R, S]^{\mathrm{S}} = \Delta_X \Leftrightarrow [K, L]_H \vee [K, L, X]_H = 0$$

Proposition [M. Hartl and T. Van der Linden, 2013]

Given a double extension F in any semi abelian category $\mathbb{X}$,

$$\begin{array}{ccc} X & \xrightarrow{f_1} & C \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow[f]{} & Z \end{array}$$

write $K_0 = \ker(f_f)$ and $K_1 = \ker(f_1)$. Then F is central if and only if

$$[K_0, K_1]_H \vee [K_0 \wedge K_1, X]_H \vee [K_0, K_1, X]_H = 0$$

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$$\bigwedge_{\varnothing} K_i = X$$

Remark

The following conditions are equivalent where $(I_l)_l$ are arbitrary subsets of $2 := \{0, 1\}$;

(i) $$[K_0, K_1]_H \vee [K_0 \wedge K_1, X]_H \vee [K_0, K_1, X]_H = 0;$$

(ii) $$\bigvee_{I_0 \cup \cdots \cup I_k = 2, k \in \mathbb{N}*} \Big[\bigwedge_{i \in I_0} K_i, \cdots , \bigwedge_{i \in I_k} K_i \Big]_H = 0$$

$[K_0, K_1, K_1]_H \subseteq [K_0, K_1]_H$ " **remove duplication enlarges the object** "

$[K_0 \wedge K_1, K_1]_H \subseteq [K_0, K_1]_H$ "**commutators are monotone**'

$[K_0 \wedge K_1, X, K_0, X, K_1, X]_H \subseteq [K_0, K_1, X]_H$

Some important results

- Higgins commutators are **reduced:** if $X_i = 0$ for some i, then $[X_1, \cdots, X_n]_H = 0$;
- Higgins commutators are **symmetric:** for any permutation $\sigma \in \Sigma_n$;

$$[X_1, \cdots, X_n]_H \cong [X_{\sigma(1)}, \cdots, X_{\sigma(n)}]_H$$

- Higgins commutators are **preserved by direct images:** for $f : X \twoheadrightarrow Y$ regular epimorphism,

$$f[X_1, \cdots, X_n]_H = [f(X_1), \cdots, f(X_n)]_H$$

Proposition

Let $\mathbb{X}$ be a semi-abelian category, X and Y two objects of $\mathbb{X}$. For any subobjects A, C of X and any subobjects B, D of Y, the square

$$\begin{array}{ccc}
(A \times B) + (C \times D) & \xrightarrow{r_{A\times B, C\times D}} & (A \times B) \times (C \times D) \\
\Big\downarrow \left\langle \begin{smallmatrix} \iota_A \times \iota_B \\ \iota_C \times \iota_D \end{smallmatrix} \right\rangle & & \Big\downarrow \langle \pi_A \times \pi_C, \pi_B \times \pi_D \rangle \\
(A + C) \times (B + D) & \xrightarrow[r_{A,C} \times r_{B,D}]{} & (A \times C) \times (B \times D)
\end{array}$$

is a regular pushout.

Proof: Let us consider the following diagram:

$$
\begin{array}{ccccc}
(A\times B)\diamond(C\times D) & \xrightarrow{\tau_{A\times B,C\times D}} & (A\times B)+(C\times D) & \xrightarrow{r_{A\times B,C\times D}} & (A\times B)\times(C\times D)\\
\downarrow{\scriptstyle \bar b} & & \downarrow{\scriptstyle \left\langle \begin{smallmatrix}\iota_A\times\iota_B\\ \iota_C\times\iota_D\end{smallmatrix}\right\rangle} & & \downarrow{\scriptstyle \langle \pi_A\times\pi_C,\pi_B\times\pi_D\rangle}\\
(A\diamond C)\times(B\diamond D) & \xrightarrow{\tau_{A,C}\times\tau_{B,D}} & (A+C)\times(B+D) & \xrightarrow[r_{A,C}\times r_{B,D}]{} & (A\times C)\times(B\times D)
\end{array}
$$

- We only need to prove that the morphism $\bar b$ is a regular epimorphism. Let us consider the pair of morphisms

$$\langle 1,0\rangle\diamond\langle 1,0\rangle : A\diamond C \to (A\times B)\diamond(C\times D)$$

$$\langle 0,1\rangle\diamond\langle 0,1\rangle : B\diamond D \to (A\times B)\diamond(C\times D)$$

- By composition with $\bar b$ we obtain the following

$$
\begin{array}{ccccc}
A\diamond C & \xrightarrow{\tau_{A,C}} & A+C & \xrightarrow{r_{A,C}} & A\times C\\
\downarrow{\scriptstyle \langle 1,0\rangle\diamond\langle 1,0\rangle} & & \downarrow{\scriptstyle \langle 1,0\rangle+\langle 1,0\rangle} & & \downarrow{\scriptstyle \langle 1,0\rangle\times\langle 1,0\rangle}\\
(A\times B)\diamond(C\times D) & \xrightarrow{\tau_{A\times B,C\times D}} & (A\times B)+(C\times D) & \xrightarrow{r_{A\times B,C\times D}} & (A\times B)\times(C\times D)\\
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(A\diamond C)\times(B\diamond D) & \xrightarrow{\tau_{A,C}\times\tau_{B,D}} & (A+C)\times(B+D) & \xrightarrow[r_{A,C}\times r_{B,D}]{} & (A\times C)\times(B\times D)
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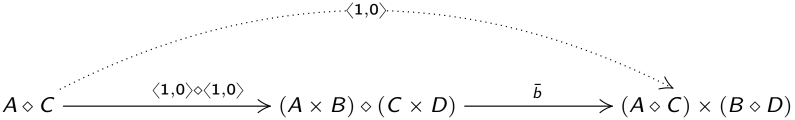
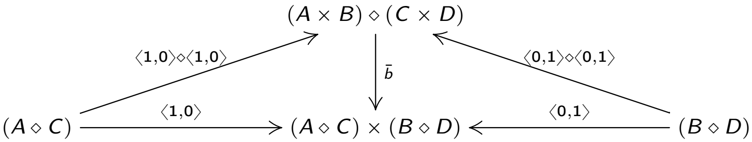


We then have:

$$
\begin{aligned}
(\tau_{A,C} \times \tau_{B,D}) \circ \bar{b} \circ \langle 1,0\rangle \diamond \langle 1,0\rangle &= \left\langle \begin{array}{c} \iota_A \times \iota_B \\ \iota_C \times \iota_D \end{array} \right\rangle \circ \tau_{A\times B, C\times D} \circ \langle 1,0\rangle \diamond \langle 1,0\rangle \\
&= \left\langle \begin{array}{c} \iota_A \times \iota_B \\ \iota_C \times \iota_D \end{array} \right\rangle \circ (\langle 1,0\rangle + \langle 1,0\rangle) \circ \tau_{A,C} \\
&= \langle 1_{A+C}, 0\rangle \circ \tau_{A,C} \\
&= (\tau_{A,C} \times \tau_{B,D}) \circ \langle 1,0\rangle
\end{aligned}
$$

Therefore, since $(\tau_{A,C} \times \tau_{B,D})$ is a monomorphism, it follows that $\bar{b} \circ \langle 1,0\rangle \diamond \langle 1,0\rangle = \langle 1,0\rangle$. Similarly, one can prove that

$$\bar{b} \circ \langle 0,1\rangle \diamond \langle 0,1\rangle = \langle 0,1\rangle$$



Therefore, $\bar{b}$ is a strong epimorphism.

Some properties

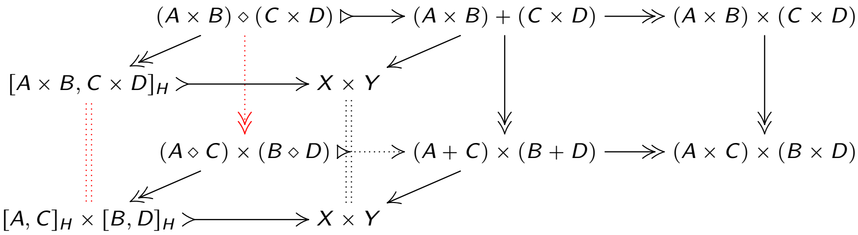
Proposition (The lower-dimensional case)

Let $\mathbb{X}$ be a semi-abelian category, X and Y two objects of $\mathbb{X}$. For any subojects A, C of X and any subobjects B, D of Y, we have:

$$[A, C]_H \times [B, D]_H = [A \times B, C \times D]_H$$

as subobjects of $X \times Y$.

Proof:



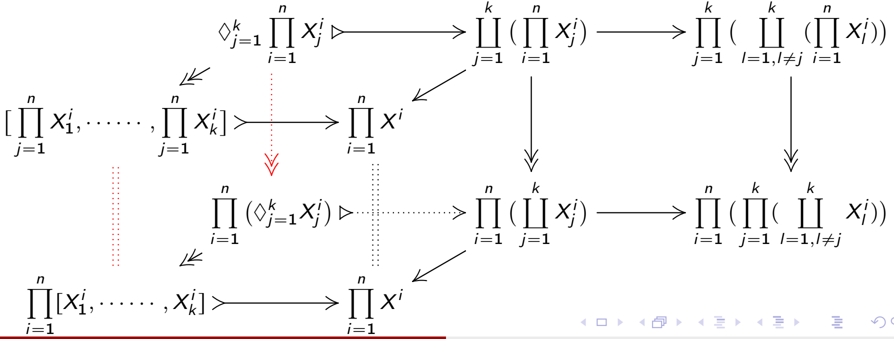
Some properties

Proposition (The higher-dimensional case)

Let $\mathbb{X}$ be a semi-abelian category, X^i, $i = 1, \cdots, n$ be objects of $\mathbb{X}$ and $x_j^i : X_j^i \to X_i$ be subobjects of X^i for $j = 1, \cdots, k$ and $i = 1, \cdots, n$. Then

$$\prod_{i=1}^{n}[X_1^i, \cdots\cdots, X_k^i]_H = [\prod_{i=1}^{n} X_1^i, \cdots\cdots, \prod_{i=1}^{n} X_k^i]_H \quad \text{as subobjects of} \quad \prod_{i=1}^{n} X^i$$

Proof:



Some properties

- We denote by Δ_A the diagonal relation $A \longrightarrow A \times A$, viewed as a subobject of $A \times A$.

Corollary

Given an object X in any semi-abelian category $\mathbb{X}$, the following properties hold:

(i) For all sub-objects $X_i \rightarrowtail X$ of X with $i = 1, \cdots, k$, we have:

$$[\Delta_{X_1}, \cdots, \Delta_{X_k}]_H = \Delta_{[X_1, \cdots, X_k]_H}$$

(ii) For all subobjects $X_i \rightarrowtail X$ of X, $i = 1, \cdots, k$, and any integer $1 \leqslant m \leqslant k$ we have:

$$[0 \times X_1, \cdots, 0 \times X_m, \Delta_{X_{m+1}}, \cdots, \Delta_{X_k}]_H = 0 \times [X_1, \cdots, X_k]_H$$

The main result

Theorem

Given an $n-$fold extension f in a semi-abelian category, write K_i for the kernel of the initial arrows $f_i : F_n \longrightarrow F_{n\setminus\{i\}}$. Then F is central if and only if the join of Higgins commutators

$$\bigvee_{I_0\cup\cdots\cup I_k=n, k\in\mathbb{N}^*} [\bigwedge_{i\in I_0} K_i, \cdots, \bigwedge_{i\in I_k} K_i]$$

vanishes.

In order to prove this theorem, it is enough to prove the following proposition:

Proposition

$$L_n[f] = \bigvee_{I_0\cup\cdots\cup I_k=n, k\in\mathbb{N}^*} [\bigwedge_{i\in I_0} K_i, \cdots, \bigwedge_{i\in I_k} K_i]_H$$

For that, we are going to use the following lemmas:

Lemma [T. Everaert, M. Gran, and T. Van der Linden, 2008]

Given any n-fold extension f in a semi-abelian category $\mathbb{X}$, we have:

$$L_n[f] = 0 \Leftrightarrow \pi_1(L_{n-1}[\mathrm{Eq}(f)] = \pi_2(L_{n-1}[\mathrm{Eq}(f)])$$

The main result

Theorem

Given an $n-$fold extension f in a semi-abelian category, write K_i for the kernel of the initial arrows $f_i : F_n \longrightarrow F_{n\setminus\{i\}}$. Then F is central if and only if the join of Higgins commutators

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For that, we are going to use the following lemmas:

Lemma [T. Everaert, M. Gran, and T. Van der Linden, 2008]

Given any n-fold extension f in a semi-abelian category $\mathbb{X}$, we have:

$$L_n[f] = 0 \Leftrightarrow \pi_1(L_{n-1}[\mathrm{Eq}(f)] = \pi_2(L_{n-1}[\mathrm{Eq}(f)])$$

Lemma

The following conditions are equivalent in any semi abelian category:

(i) For every n-fold extension $f : B \longrightarrow A$,

$$L_n[f] = 0 \Leftrightarrow \bigvee_{I_0 \cup \cdots \cup I_k = n, k \in \mathbb{N}^*} [\bigwedge_{i \in I_0} K_i, \cdots, \bigwedge_{i \in I_k} K_i]_H = 0 \qquad (A)$$

(ii) For every n-fold extension $f : B \longrightarrow A$,

$$L_n[f] = \bigvee_{I_0 \cup \cdots \cup I_k = n, k \in \mathbb{N}^*} [\bigwedge_{i \in I_0} K_i, \cdots, \bigwedge_{i \in I_k} K_i]_H \qquad (B)$$

Proof of the proposition by induction on n

• For $n = 0$, $f = A$ and $L_0[f] = [A, A]_H$

• Now let us assume that the result holds for $(n-1)$-fold extensions. Let $f : B \to A$ be an n-fold extension.

- $\mathrm{Eq}(f)$ is an $(n-1)$-fold extension so that

$$L_{n-1}[\mathrm{Eq}(f)] \quad = \bigvee_{I_0\cup\cdots\cup I_k=n-1, k\in\mathbb{N}^*} \Big[\bigcap_{i\in I_0} K[\mathrm{Eq}(f)_i], \cdots, \bigcap_{i\in I_k} K[\mathrm{Eq}(f)_i]\Big]_H$$

Proposition [M. Hartl and T. Van der Linden, 2013]

Commutators satisfy a distribution rule with respect to joins:

$$\big[X_1, \cdots, X_n, A_1 \vee \cdots \vee A_n\big]_H = \bigvee_{1\leqslant k\leqslant m, 1\leqslant i_1<\cdots<i_k\leqslant m} \big[X_1, \cdots, X_n, A_{i_1}, \cdots, A_{i_m}\big]_H$$

With all the above results, we proved that the following conditions are equivalent

(i) $L_n[F] = 0$;

(ii) $\pi_1 L_{n-1}[\mathrm{Eq}(f)] = \pi_2 L_{n-1}[\mathrm{Eq}(f)]$;

(iii) $\displaystyle\bigvee_{I_0\cup\cdots\cup I_m=n, m\in\mathbb{N}^*} \Big[\bigwedge_{i\in I_0} K_i, \cdots, \bigwedge_{i\in I_m} K_i\Big]_H = 0$

F is central if and only if the join of Higgins commutators

$$\bigvee_{I_0 \cup \cdots \cup I_k = n, k \in \mathbb{N}^*} \Big[\bigwedge_{i \in I_0} K_i, \cdots, \bigwedge_{i \in I_k} K_i \Big]_H = 0$$

Corollary

In any semi-abelian monadic category $\mathbb{X}$, for any n-presentation F of an object Z,

$$H_{n+1}(Z, \mathrm{Ab}(\mathbb{X})) \cong \frac{[F_n, F_n]_H \wedge \bigwedge_{i \in n} \ker(f_i)}{\bigvee_{I_0 \cup \cdots \cup I_k = n, k \in \mathbb{N}^*} \big[\bigwedge_{i \in I_0} K_i, \cdots, \bigwedge_{i \in I_k} K_i \big]}$$

Some perspectives

As mentioned by **M.Hartl**, the combinatorial computation of the generators of the Higgins commutator $[X_1, \cdots, X_n]_H$ as a normal subobject of $X_1 \vee \cdots \vee X_n$, for $n \geqslant 4$ in any semi-abelian category is still an open problem.

Generators of Higgins commutators in Grp [B. Loiseau]

$[X_1, \cdots, X_n]_H$ is generated as a subgroup by all nested commutators (with arbitrary bracketing) of elements $x_1 \in X_{k_1}, \cdots, x_m \in X_{k_m}$ such that $\{k_1, \cdots, k_m\} = \{1, \cdots, n\}$

In the future, I would like to:

1. Describe all generators of $[X_1, \cdots, X_n]_H$, as a normal subobject of $X_1 \vee \cdots \vee X_n$ in any semi-abelian variety.

2. Study the difference between the n-fold Higgins commutator $[X_1, \cdots, X_n]_H$ of normal subobjects and the normalisation of the Bulatov commutator of their denormalisations.

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[6] D. Rodelo and T. Van der Linden, *Higher central extensions via commutators*, Theory Appl. Categ. 27 (2012), no. 9, 189-209.

Thank you!

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