Limits in categories of Vietoris coalgebras

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Collaboration with Renato Neves and Pedro Nora.

July 10, 2018

Motivation

Question

For "the" Vietoris functor $V\colon \mathbf{Top} \to \mathbf{Top}$, is the category $\mathrm{CoAlg}(V)$ of coalgebras for V complete (or has at least a terminal object)?

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- For a functor $F\colon \mathbf{C} \to \mathbf{C}$, a coalgebra

$$\begin{array}{ccc} FX & \qquad & FY \\ c \uparrow & & \uparrow d \\ X & & Y \end{array}$$

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- Think of $V\colon \mathbf{Top} \to \mathbf{Top}$ as a "topological powerset functor".

Theorem

The canonical forgetful functor $\mathrm{CoAlg}(F) \to \mathbf{C}$ *creates colimits.*

What about limits?

Theorem

$\mathrm{CoAlg}(F) \to \mathbf{C}$ *creates those limits which* $F \colon \mathbf{C} \to \mathbf{C}$ *preserves.*

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The powerset functor $P\colon \mathbf{Set} \to \mathbf{Set}$ does not admit a terminal coalgebra.

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Hence, one might expect that $\mathrm{CoAlg}(V)$ is also not complete ...

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Hence, one might expect that $\mathrm{CoAlg}(V)$ is also not complete ...

... however, $V\colon \mathbf{Top} \to \mathbf{Top}$ does admit a terminal coalgebra; so who knows ... (we don't).

What follows is what we (believe to) know ....

A primer on limits in categories of coalgebras

Some well-known results

Theorem

If the $\mathbf{C}$ *has and* $F\colon \mathbf{C} \to \mathbf{C}$ *preserves the limit* L *of the diagram*

$$1 \longleftarrow F1 \longleftarrow FF1 \longleftarrow \dots,$$

then the canonical isomorphism $L \to FL$ *is a terminal* F*-coalgebra.*

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If $\mathbf{C}$ *is cocomplete with finite limits and* $\mathbf{C}$ *has and* $F\colon \mathbf{C} \to \mathbf{C}$ *preserves limits of countable chains, then* $F\colon \mathbf{C} \to \mathbf{C}$ *is a covarietor.*

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Theorem

Let F be a covarietor over a complete category. If $\mathrm{CoAlg}(F)$ has equalisers then $\mathrm{CoAlg}(F)$ is complete.

Cocompleteness implies completeness

Theorem

Let $F \colon \mathbf{C} \to \mathbf{C}$ be an endofunctor on a cocomplete category $\mathbf{C}$ and let I be a small category. If $\mathbf{C}$ is $(E, \mathcal{M})$-structured for cones for I, $\mathcal{M}$-wellpowered and F sends cones in $\mathcal{M}$ to cones in $\mathcal{M}$, then $\mathrm{CoAlg}(F)$ *has limits of shape I.*

Proof.

Verify the Solution Set Condition for $\Delta \colon \mathrm{CoAlg}(F) \longrightarrow \mathrm{CoAlg}(F)^I$. $\square$

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Corollary

If $F\colon \mathbf{Set} \to \mathbf{Set}$ preserves monocones of a certain type, then the category $\mathrm{CoAlg}(F)$ has limits of the same type.

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Theorem

Let $F: \mathbf{C} \to \mathbf{C}$ be an endofunctor on a cocomplete category $\mathbf{C}$ and let I be a small category. If $\mathbf{C}$ is $(E, \mathcal{M})$-structured for cones for I, $\mathcal{M}$-wellpowered and F sends cones in $\mathcal{M}$ to cones in $\mathcal{M}$, then $\mathrm{CoAlg}(F)$ has limits of shape I.

Corollary

If $F: \mathbf{Set} \to \mathbf{Set}$ preserves monocones of a certain type, then the category $\mathrm{CoAlg}(F)$ has limits of the same type.

Corollary

If $F: \mathbf{Top} \to \mathbf{Top}$ preserves either small monocones or small initial monocones of a certain type, then the category $\mathrm{CoAlg}(F)$ has limits of the same type.

Vietoris functors

Vietoris functors on topological spaces

"Das Orginal"

For a compact Hausdorff space X, the classic Vietoris space VX consists of the set of all closed subsets of X

$$VX = \{K \subseteq X \mid K \text{ is closed}\}$$

equipped with the "hit-and-miss topology" generated by the subbasis of sets of the form (where $U \subseteq X$ is open)

$$U^{\Diamond} = \{A \in VX \mid A \cap U \neq \varnothing\} \qquad (\text{"}A \text{ hits } U\text{"}),$$
$$U^{\Box} = \{A \in VX \mid A \cap U^{\complement} = \varnothing\} \qquad (\text{"}A \text{ misses } U^{\complement}\text{"}).$$

We obtain $V\colon \mathbf{CompHaus} \to \mathbf{CompHaus}$.

Leopold Vietoris (1922). "Bereiche zweiter Ordnung". In: *Monatshefte für Mathematik und Physik* **32**.(1), pp. 258–280.

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This definition can be generalised to arbitrary topological spaces ... but does not always define a functor!!

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We consider here the following two variants on **Top**:

- lower Vietoris: closed subsets, but only "miss topology".

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We consider here the following two variants on **Top**:

- lower Vietoris: closed subsets, but only "miss topology".
- compact Vietoris: compact subsets, "hit-and-miss topology".

Vietoris functors more systematic (?)

Covariant presheafs

Consider, for a topological space X: $\quad X \longmapsto \; 2^X$

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- The convergence 2^X can be split into a function $\mu\colon U(2^x) \to 2^x$ and the order relation $\subseteq$: $\mathfrak{p} \to A \iff \mu(\mathfrak{p}) \subseteq A$.

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Restricting to (stably) compact spaces

The lower Vietoris functor restricts to $V\colon$ **StablyComp** $\to$ **StablyComp**

(those topological spaces X where the convergence splits "nicely" into a compact Hausdorff topology $\alpha\colon UX \to X$ and a partial order $\leq$ on X)

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and can be transferred along the adjunction above which leads to the classic Vietoris functor $V\colon \mathbf{CompHaus} \to \mathbf{CompHaus}$.

General properties of Vietoris functors on **Top**

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- *For every lower (compact) Vietoris polynomial functor* $F : \mathbf{Top} \to \mathbf{Top}$ *the category* $\mathrm{CoAlg}(F)$ *has codirected limits (of Hausdorff spaces).*

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Remark

None of the Vietoris functors preserves codirected limits in **Top**.

What is known (to us) on subcategories of **Top**

For $V \colon \mathbf{BooSp} \to \mathbf{BooSp}$, $\mathrm{CoAlg}(V)$ is complete . . .

Samson Abramsky (2005). "A Cook's Tour of the Finitary Non-Well-Founded Sets". In: *We Will Show Them! Essays in Honour of Dov Gabbay*. Ed. by S. Artemov, H. Barringer, and A. A. Garcez. London: College Publications, pp. 1–18.

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... and even better: $\mathrm{CoAlg}(V)^{\mathrm{op}} \simeq \mathbf{BAO}$ is a finitary variety.

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The compact Vietoris functor $V\colon \mathbf{Haus} \to \mathbf{Haus}$ preserves codirected limits. Hence, for all compact Vietoris polynomial functors $F\colon \mathbf{Haus} \to \mathbf{Haus}$, $\mathrm{CoAlg}(F)$ is complete.

Phillip Zenor (1970). "On the completeness of the space of compact subsets". In: *Proceedings of the American Mathematical Society* **26**.(1), pp. 190–192.

Cofiltered limits in **CompHaus**

Theorem

Let $D\colon I \to \mathbf{CompHaus}$ *be a cofiltered diagram. Then* $(p_i\colon L \to D(i))_{i\in I}$ *for* D *is a limit cone if and only if*

1. $(p_i\colon L \to D(i))_{i\in I}$ *is mono and,*
2. *for every* $i \in I$*:* $\bigcap_{j\to i} \operatorname{im} D(j \to i) = \operatorname{im} p_i$*;*

   That is, "the image of each p_i *is as large as possible".*

Nicolas Bourbaki (1942). *Éléments de mathématique. 3. Pt. 1: Les structures fondamentales de l'analyse. Livre 3: Topologie générale.* Paris: Hermann & Cie.

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Compare with:

Let $D \colon I \to \mathbf{Set}$ be a filtered diagram. Then $(c_i \colon D(i) \to C)_{i \in I}$ is a colimit of D if and only if

1. $(c_i \colon D(i) \to C)_{i \in I}$ is epi and,
2. for all $i \in I$ and $x, y \in D(i)$,
$$c_i(x) = c_i(y) \iff \exists (i \xrightarrow{k} j) \in I \,.\, D(k)(x) = D(k)(y);$$
   that is, "the kernel of each c_i is as small as possible".

Vietoris coalgebras of stably compact spaces

Theorem

All Vietoris polynomial functors $F\colon \mathbf{StablyComp} \to \mathbf{StablyComp}$ *preserve cofiltered limits. Hence,* $\mathrm{CoAlg}(F)$ *is complete.*

Proof.

Use

$\square$

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- limit in **StablyComp** = limit in **CompHaus** + initial monocone,
- the previous characterisation of cofiltered limits,
- initial monocone in **StablyComp** = initial monocone in **Top**, and
- the fact that $V\colon \mathbf{StablyComp} \to \mathbf{StablyComp}$ preserves initial monocones.

□

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All Vietoris polynomial functors $F\colon \mathbf{StablyComp} \to \mathbf{StablyComp}$ preserve cofiltered limits. Hence, $\mathrm{CoAlg}(F)$ is complete.

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Every lower Vietoris polynomial functor in **Top** *that can be restricted to* **StablyComp** *admits a terminal coalgebra. In particular, $\mathrm{CoAlg}(V)$ has a terminal object.*

Proof.

Use that $\mathbf{StablyComp} \hookrightarrow \mathbf{Top}$ is closed under limits and

$$1 \longleftarrow F1 \longleftarrow FF1 \longleftarrow \dots$$

lives in **StablyComp**. □

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Every lower Vietoris polynomial functor in $\mathbf{Top}$ *that can be restricted to* $\mathbf{StablyComp}$ *admits a terminal coalgebra. In particular,* $\mathrm{CoAlg}(V)$ *has a terminal object.*

Corollary

The lower Vietoris functor $V\colon \mathbf{Top} \to \mathbf{Top}$ *admits a terminal coalgebra.*

Vietoris coalgebras via duality theory

Remark

For $V\colon \mathbf{BooSp} \to \mathbf{BooSp}$, the dual equivalence

$$\mathsf{CoAlg}(V) \simeq \mathbf{BAO}^{\mathrm{op}}$$

follows immediately from Halmos duality:

$$\mathbf{BooSp}_{\mathbb{V}} \simeq \mathbf{BA}^{\mathrm{op}}_{\perp,\vee}:$$

Duality theory for coalgebras on Boolean spaces

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- Coalgebra $X \to VX \quad = \quad$ endomorphism in $\mathbf{BooSp}_{\mathbb{V}}$.

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For $V\colon \mathbf{BooSp} \to \mathbf{BooSp}$, the dual equivalence

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Objective

Develop a similar duality theory for $\mathbf{StablyComp}_{\mathbb{V}}$.

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Theorem

For $V\colon \mathbf{PosComp} \to \mathbf{PosComp}$, $\mathsf{CoAlg}(V)^{\mathrm{op}}$ *is an* $\aleph_1$*-ary quasivariety.*

Enriched Stone-type dualities

Theorem

Consider the quantale $[0,1]$ *ordered by the "greater or equal" relation* $\geqslant$ *and tensor product* $\oplus$ *given by truncated addition:*

$$u \oplus v = \min(1, u + v).$$

Then $[0,1]$-**Cat** *is the category of "bounded-by-1" metric spaces and non-expansive maps.*

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$$\mathbf{StablyComp}_{\mathbb{V}} \xrightarrow{C(-,[0,1])} \mathrm{LaxMon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$$

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  Thanks to Adriana Balan and

Anders Kock (1972). "Strong functors and monoidal monads". In: *Archiv der Mathematik* **23**.(1), pp. 113–120.

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$$\mathbf{StablyComp} \xrightarrow{C(-,[0,1])} \mathsf{Mon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}} \text{ is fully faithful.}$$

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$\mathrm{Mon}([0,1]\text{-}\mathbf{FinSup})$ *is an* $\aleph_1$*-ary quasivariety (and fully embeds into a finitary variety).*

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Mon([0, 1]-**FinSup**) *is an* $\aleph_1$*-ary quasivariety (and fully embeds into a finitary variety).*

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\begin{tikzcd}
\mathbf{PosComp}_{\mathbb{V}}^{\mathrm{op}} \arrow[r, "\sim"] & \mathbf{B}_{[0,1]} & \\
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$[0,1]$ *is $\aleph_1$-copresentable in* $\mathbf{PosComp}$, *hence* $\mathbf{A}_{[0,1]}$ *is an $\aleph_1$-ary quasivariety.*

CoAlg(V) as a quasivariety

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Theorem

For $V : \mathbf{PosComp} \to \mathbf{PosComp}$, $\mathsf{CoAlg}(V)^{\mathrm{op}}$ *is an* $\aleph_1$*-ary quasivariety.*

Proof.

Consider the algebraic theory of $\mathbf{A}_{[0,1]}$ augmented

- by one unary operation symbol and
- by those equations which express that the corresponding operation is a finitely cocontinuous $[0,1]$-functors laxly preserving the monoid structure.

□

$\mathsf{CoAlg}(V)$ as a quasivariety

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Theorem

The $\aleph_1$*-copresentable objects in* **PosComp** *are precisely the "generalised metrisable" partially ordered compact space (i.e. induced by a* $[0,1]$*-category).*