On Suborbifolds

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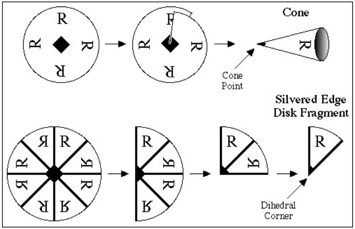
Outline

Orbifolds - Informally

An n-dimensional orbifold is a paracompact space that is locally homeomorphic to the quotient of $\mathbb{R}^n$ by a smooth action of a finite group.



Orbifolds as Groupoids

An **orbifold groupoid** is a **proper étale groupoid** in the category of smooth manifolds,

$$\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \xrightarrow{m} \mathcal{G}_1 \xrightarrow{\text{inv}} \mathcal{G}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathcal{G}_0 \quad (u : \mathcal{G}_0 \to \mathcal{G}_1)$$

- Each structure map is a local diffeomorphism.
- The map $(s, t) \colon \mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0$ is proper (closed with compact fibers).

Examples: Action Groupoids

- Let G be a finite group acting on an open $U \subseteq \mathbb{R}^n$, then the action groupoid $G \ltimes U$,

$$G \times G \times U \xrightarrow{m} G \times U \xrightarrow[(g,u)\mapsto(g^{-1},g\cdot u)]{\text{inv}} G \times U \underset{t=a}{\overset{s=\pi_2}{\rightrightarrows}} U \quad (u: U \to G \times U)$$

  is proper and étale. This is the prototype for an **orbifold atlas chart**.

- The action groupoid $G \ltimes M$ for any properly discontinuous action of a group G on a manifold M. These orbifolds are called **global quotients**.

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  is proper and étale. This is the prototype for an **orbifold atlas chart**.
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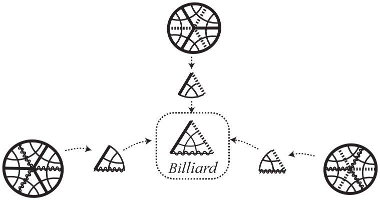
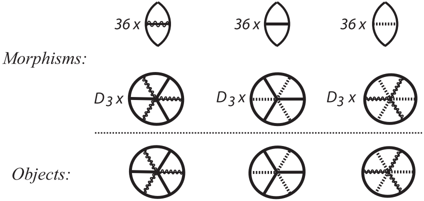
Example 2: The Triangular Billiard

- The quotient space



- The groupoid



Properties of Orbifold Groupoids

- All isotropy groups

$$\mathcal{G}_x = s^{-1}(x) \cap t^{-1}(x)$$

  (for $x \in \mathcal{G}_0$) are finite and discrete;
- Each point $x \in \mathcal{G}_0$ has a neighbourhood basis of opens U_x with the property that

$$\mathcal{G}|_{U_x} \cong \mathcal{G}_x \ltimes U_x.$$

  We call such neighbourhoods **chart neighbourhoods**.

Essential Equivalences

- A morphism $f\colon \mathcal{G} \to \mathcal{H}$ is an **essential equivalence** when it is essentially surjective and fully faithful.
- It is **essentially surjective** when $\mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \longrightarrow \mathcal{H}_0$ in

$$\begin{array}{ccccc} \mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 & \longrightarrow & \mathcal{H}_1 & \xrightarrow{t} & \mathcal{H}_0 \\ \downarrow & & \downarrow s & & \\ \mathcal{G}_0 & \xrightarrow[f_0]{} & \mathcal{H}_0 & & \end{array}$$

  is a **surjective submersion**.
- The morphism $f\colon \mathcal{G} \to \mathcal{H}$ is **fully faithful** when

$$\begin{array}{ccc} G_1 & \xrightarrow{\phi} & H_1 \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ G_0 \times G_0 & \xrightarrow{\phi\times\phi} & H_0 \times H_0 \end{array}$$

  is a **pullback**.

Morita Equivalence

- Two orbigroupoids $\mathcal{G}$ and $\mathcal{H}$ are called **Morita equivalent** if there exists a third orbigroupoid $\mathcal{K}$ with essential equivalences

$$\mathcal{G} \xleftarrow{\varphi} \mathcal{K} \xrightarrow{\psi} \mathcal{H}.$$

- This is an equivalence relation on groupoids, because essential equivalences of topological groupoids are stable under weak pullbacks (iso-comma-squares).

Maps Between Orbifolds as Spans

The bicategory of orbifolds is the bicategory of fractions of the 2-category of orbifold groupoids with respect to the class of essential equivalences. Maps can be described as spans,

$$\mathcal{G} \xleftarrow{\upsilon} \mathcal{K} \xrightarrow{\varphi} \mathcal{H}$$

where υ is an essential equivalence.

Equivalently, maps between orbifolds can be described as Hilsum-Skandalis bimodules.

Maps Between Orbifolds as Modules

Definition

A **Hilsum-Skandalis bimodule** $M \colon \mathcal{G} \twoheadrightarrow \mathcal{H}$ consists of a manifold

$$\mathcal{G}_0 \xleftarrow{\;q\;} M \xrightarrow{\;p\;} \mathcal{H}_0,$$

with $q \colon M \to \mathcal{G}_0$ a surjective submersion, and groupoid actions:

- $\mathcal{G}_1 \times_{\mathcal{G}_0} M \to M$ over $\mathcal{G}_0$: when $t(g) = q(m)$, $q(g \cdot m) = s(g)$ and $p(g \cdot m) = p(m)$;
- $M \times_{\mathcal{H}_0} \mathcal{H}_1 \to M$ over $\mathcal{H}_0$: when $s(h) = p(m)$, $p(m \cdot h) = t(h)$ and $q(m \cdot h) = q(m)$;

such that $(g \cdot m) \cdot h = g \cdot (m \cdot h)$ and $M \times_{\mathcal{G}_0} M \cong \mathcal{H}_1 \times_{\mathcal{H}_0} M$.

Correspondence 1: from Groupoids to Modules

Given a generalized map,

$$\mathcal{G} \xleftarrow{\upsilon} \mathcal{K} \xrightarrow{\psi} \mathcal{H}$$

one forms the module

$$M_{\mathcal{K}} = (\mathcal{G}_1 \times_{\mathcal{K}_0} \mathcal{H}_1)/\mathcal{K}_1$$

with $(g\upsilon(k)^{-1}, g) \sim (g, \upsilon(k)h)$ and

$$\mathcal{G}_0 \xleftarrow{t\pi_1} M_{\mathcal{K}} \xrightarrow{t\pi_2} \mathcal{H}_0$$

and actions defined by composition in the respective groupoids.

Correspondence 2: from Modules to Groupoids

Given a right $\mathcal{G}$- left $\mathcal{H}$-module,

$$\mathcal{G}_0 \xleftarrow{\;q\;} M \xrightarrow{\;p\;} \mathcal{H}_0$$

we obtain a generalized map

$$\begin{array}{ccccc}
\mathcal{G}_1 & \xleftarrow{\;\pi_1\;} & \mathcal{G}_1 \times_{\mathcal{G}_0} M \times_{\mathcal{H}_0} \mathcal{H}_1 & \xrightarrow{\;\pi_3\;} & \mathcal{H}_1 \\
s \downarrow\downarrow t & & s(g,m,h)=m \downarrow\downarrow t(g,m,h)=gmh^{-1} & & s \downarrow\downarrow t \\
\mathcal{G}_0 & \xleftarrow[\;q\;]{} & M & \xrightarrow[\;p\;]{} & \mathcal{H}_0
\end{array}$$

Geometric Embeddings

- A geometric morphism $\mathcal{E} \underset{\varphi_*}{\overset{\varphi^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{F}$ is an **embedding** if φ_* is full and faithful.
- If the geometric morphism is induced by a groupoid morphism $\varphi\colon \mathcal{G} \to \mathcal{H}$, $\varphi_*\colon \mathrm{Sh}(\mathcal{G}) \to \mathrm{Sh}(\mathcal{H})$ is full and faithful if and only if φ is.

Thurston's Effective Suborbifolds

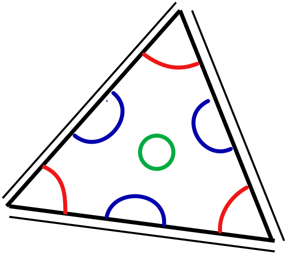
Definition (Thurston)

A d-dimensional suborbifold Q_1 of Q_2 is given by a subspace $X_{Q_1} \subset X_{Q_2}$ locally modeled on $\mathbb{R}^d$ modulo the induced actions of the local groups of Q_2 on invariant d-dimensional subspaces.

Remark

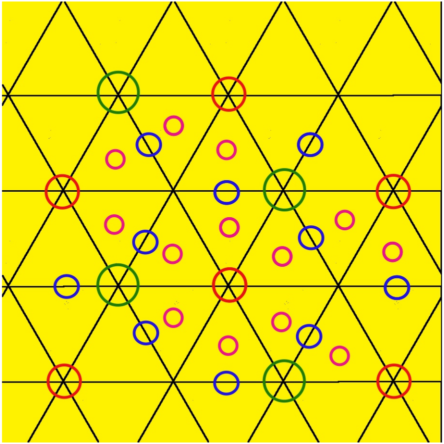
- This means that for any chart U, the preimage of the subspace X_{Q_1} under $U \to U/G_U \hookrightarrow X_{Q_2}$ needs to be a submanifold of U, invariant under the action of the structure group G_U.
- Furthermore, Thurston only takes the effective part of the actions on the submanifold to obtain the structure group for the suborbifold.

Example: The Triangular Billiard



All 1-dimensional closed suborbifolds up to isotopy according to Thurston

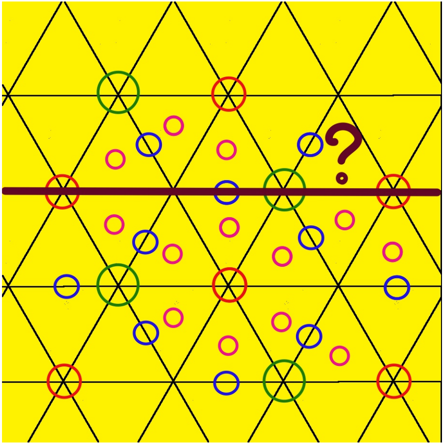
A Universal Covering of the Triangular Billiard



All Thurston suborbifolds give rise to submanifolds of the universal covering.

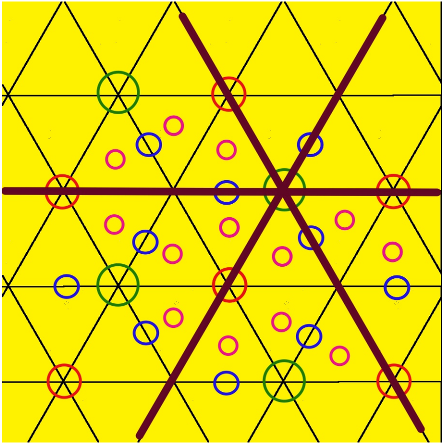
A Universal Covering of the Triangular Billiard

Are there any other suborbifolds?



A Universal Covering of the Triangular Billiard

NO!



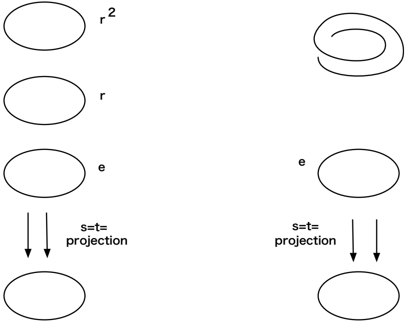
Thurston Suborbifolds

Proposition

Thurston suborbifolds of $\mathcal{G}$ *are the effective/reduced parts of geometric embeddings* $Sh(\mathcal{H}) \to Sh(\mathcal{G})$.

Other Geometric Suborbifolds

- The group $\mathbb{Z}/3$ acts by rotation about the central axis on both the solid torus and the solid Klein bottle.
- In both cases the central axis gives a suborbifold with a completely ineffective $\mathbb{Z}/3$-action, whose reduced orbifold is the circle S^1.
- However, the ineffective orbifolds are not Morita equivalent:

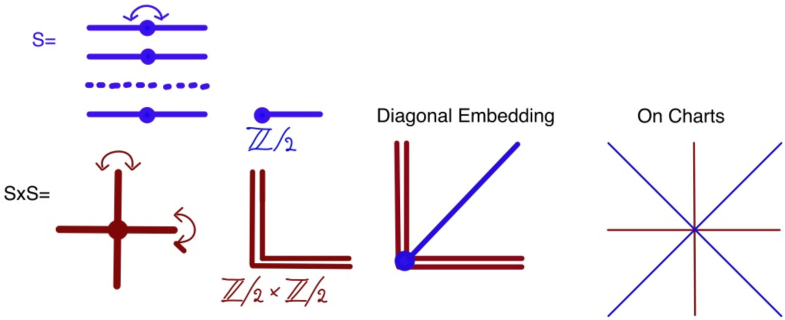


Motivation

In *Orbifolds and Stringy Topology*, Adem, Leida and Ruan give several reasons for a more general concept of suborbifold:

- Suborbifolds need to have properties similar to those of submanifolds in terms of homotopy theory: we want all suborbifolds for which we could construct a Poincaré dual.
- The diagonal $\Delta_{\mathcal{G}} \colon \mathcal{G} \to \mathcal{G} \times \mathcal{G}$ needs to be a suborbifold.
- The connected components of the inertia orbifold $\Lambda\mathcal{G}$ need to be suborbifolds of $\mathcal{G}$.

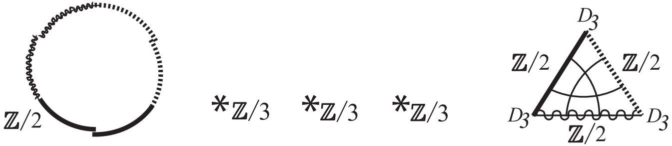
An Example: the Diagonal



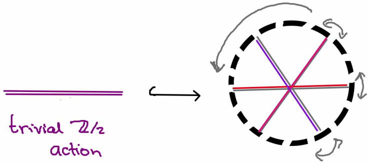
The Inertia Orbifold of the Triangular Billiard

Let $(X, \mathfrak{U})$ be an orbifold. Its **inertia orbifold** is defined as follows

- Its charts are the fixed point sets of the actions of the elements of the structure groups on the charts of X.
- The embedding modules between them are formed by restrictions of the modules of $(\mathfrak{U})$.



New Embeddings



The right adjoint inclusion functor is in general only faithful.

Cho, Hong and Shin's Definition

Definition (Cho-Hong-Shin, 2013)

A homomorphism of orbifold groupoids $\varphi\colon \mathcal{G} \to \mathcal{H}$ is an **embedding** if:

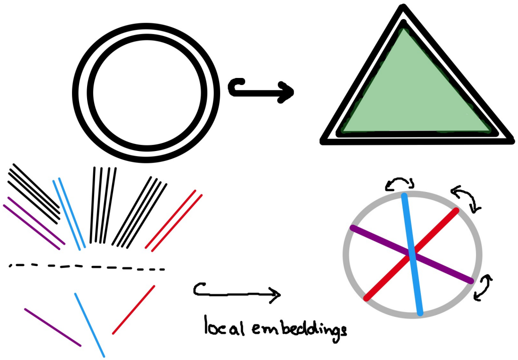
1. $\varphi_0\colon \mathcal{G}_0 \to \mathcal{H}_0$ is an immersion
2. Let $y \in \mathrm{Im}\,(\varphi_0) \subset \mathcal{H}_0$ and V_y a chart neighbourhood, $\mathcal{H}|_{V_y} \cong \mathcal{H}_y \ltimes V_y$. Then, the $\mathcal{G}$-action on $\varphi_0^{-1}(y)$ is transitive, and there exists an open chart neighborhood $U_x \subset \mathcal{G}_0$ for each $x \in \varphi_0^{-1}(y)$ such that

$$\mathcal{G}|_{\varphi_0^{-1}(V_y)} \cong \mathcal{H}_y \ltimes (\mathcal{H}_y \times_{\mathcal{G}_x} U_x)$$

3. $|\varphi|\colon |\mathcal{G}| \to |\mathcal{H}|$ is proper and injective.

$\mathcal{G}$ together with φ is called an orbifold embedding of $\mathcal{H}$.

Example: The Inertia Circle of the Triangular Billiard



Some Concerns

- This definition seems to favour only one type of representation of the suborbifold determined by the representation of the larger orbifold.
- The definition seems to be unnecessarily complicated.

Orbifold Embeddings - A New Definition

Definition (PST)

A groupoid homomorphism $\varphi\colon \mathcal{G} \to \mathcal{H}$ between orbifold groupoids is an orbifold embedding if

- it is faithful;
- $\varphi_0\colon \mathcal{G}_0 \to \mathcal{H}_0$ is an immersion;
- the induced map $\bar{\varphi}\colon \mathcal{G}_0/\mathcal{G}_1 \to \mathcal{H}_0/\mathcal{H}_1$ is injective.

Connection via Hilsum-Skandalis Maps

The connection with the CHS-definition of suborbifold can be made through the Hilsum-Skandalis map presentation of morphisms. Given $\varphi\colon \mathcal{G} \to \mathcal{H}$, the corresponding Hilsum-Skandalis module is

$$\mathcal{G}_0 \xleftarrow{\ \pi_1\ } M_\varphi = \mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \xrightarrow{\ t\pi_2\ } \mathcal{H}_0$$

Connection via Hilsum-Skandalis Maps

For $x \in \mathcal{G}_0$ and $y \in \mathcal{H}_0$,

$$M_\varphi = \mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \supseteq \pi_1^{-1}(x) \cap (t\pi_2)^{-1}(y)$$

$$x \in \mathcal{G}_0 \xleftarrow{\ \pi_1\ } M_\varphi \xrightarrow{\ t\pi_2\ } \mathcal{H}_0 \ni y$$

Note that $\pi_1^{-1}(x) \cap (t\pi_2)^{-1}(y) \neq \emptyset$ if and only if $\varphi(x)$ and y are in the same orbit in $\mathcal{H}$.

Connection via Hilsum-Skandalis Maps

- In that case, there are chart neighbourhoods U_x and V_y respectively,

$$\begin{array}{ccc} & M_\varphi = \mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 \supseteq \pi_1^{-1}(U_x) \cap (t\pi_2)^{-1}(V_y) & \\ {}^{\pi_1}\swarrow & & \searrow^{t\pi_2} \\ U_x \subseteq \mathcal{G}_0 & & \mathcal{H}_0 \supseteq V_y \end{array}$$

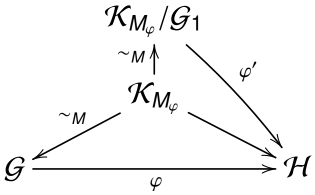
  such that

$$\pi_1^{-1}(U_x) \cap (t\pi_2)^{-1}(V_y) \cong H_y \times U_x$$

- H_y acts freely and transitively and G_x acts freely on this subspace.

Hilsum-Skandalis Embeddings

- Now consider the groupoid $\mathcal{K}_{M_\varphi}$,
$$\mathcal{G}_1 \times_{\mathcal{G}_0} M_\varphi \times_{\mathcal{H}_0} \mathcal{H}_1 \rightrightarrows M_\varphi.$$
- We can mod out by the $\mathcal{G}_1$ action since it is free, and obtain a groupoid $\mathcal{K}_{M_\varphi}/\mathcal{G}_1$ which restricted to $(\pi_1^{-1}(U_x) \cap (t\pi_2)^{-1}(V_y))/\mathcal{G}_1$ is precisely $H_y \ltimes (H_y \times_{G_x} U_x)$.
- So we have the following diagram



and whenever φ is an embedding by our definition, φ' is one according to [CHS].

Embeddings

Proposition

Each orbifold embedding according to our definition is Morita equivalent to one according to [CHS].

Hilsum-Skandalis Embeddings

Definition (PST)

A Hilsum-Skandalis map $M\colon \mathcal{G} \nrightarrow \mathcal{H}$ between orbifold groupoids is an orbifold embedding if

- The map $M \to \mathcal{H}_0$ is an immersion;
- The action of $\mathcal{G}$ on M is free.

Main Theorem

Theorem

The following are equivalent for a geometric morphism

$$Sh(\mathcal{G}) \underset{\varphi_*}{\overset{\varphi^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} Sh(\mathcal{H}) :$$

- $\varphi_* \cong \psi_* \upsilon^*$ *for a span of groupoids* $\mathcal{G} \overset{\upsilon}{\leftarrow} \mathcal{K} \overset{\psi}{\rightarrow} \mathcal{H}$ *where* υ *is an essential equivalence and* ψ *is an embedding.*
- $\varphi_* \cong M_*$ *for a Hilbert-Skandalis module* $M : \mathcal{G} \twoheadrightarrow \mathcal{H}$ *which is an embedding of orbifold groupoids.*
- φ_* *is faithful, induces an injective map of quotient spaces, and induces local geometric embeddings* $(Sh(U), C^\infty(U)) \to (Sh(\mathcal{H}_0), C^\infty(\mathcal{H}_0))$ *for a covering of* $\mathcal{G}_0$ *by open subsets* $U \in \mathcal{U}$.

Further Comments and Results

- The CHS-notion of suborbifold has been adjusted to effective orbifolds and atlases by Borzellino and Brunsden.
- The BB-notion generalizes to one of suborbifolds in terms of our atlases that matches the notion presented here.
- The various descriptions of suborbifolds given in terms of groupoid morphisms, Hilsum-Skandalis modules, and atlases lend themselves to the consideration of various additional properties that are not as easily expressed in terms of geometric morphisms.

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