Groupoids Associated to Inverse Categories

Category Theory 2018 at University of the Azores

Darien DeWolf
St. Francis Xavier University
Antigonish, Nova Scotia, Canada

July 13, 2018

Inverse Categories

Inverse Categories

The multi-object of inverse semigroups.

Inverse Categories

The multi-object of inverse semigroups.

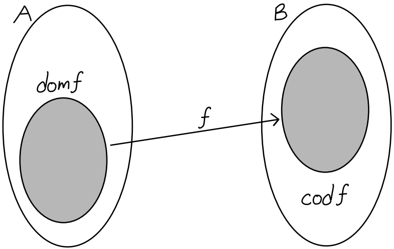
Defining property of inverse categories:
For each arrow $f : A \to B$, there is a unique arrow $f^\circ : B \to A$ such that $ff^\circ f = f$ and $f^\circ ff^\circ = f^\circ$.

Inverse Categories

The multi-object of inverse semigroups.

Defining property of inverse categories:
For each arrow $f : A \to B$, there is a unique arrow $f^\circ : B \to A$ such that $ff^\circ f = f$ and $f^\circ ff^\circ = f^\circ$.

Prototypical example: sets and partial bijections, using relation composition:

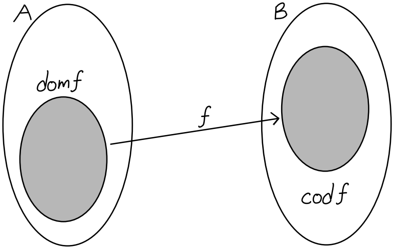


Inverse Categories

The multi-object of inverse semigroups.

Defining property of inverse categories:
For each arrow $f : A \to B$, there is a unique arrow $f^\circ : B \to A$ such that $ff^\circ f = f$ and $f^\circ ff^\circ = f^\circ$.

Prototypical example: sets and partial bijections, using relation composition:



Notation: $\mathrm{dom} f = \overline{f} = f^\circ f$, convenient to think of $\overline{f}$ as the identity on the domain of f (where f is defined).

Groupoids Associated to Inverse Categories

Groupoids Associated to Inverse Categories

As a generalization of the Ehresmann-Schein-Nambooripad Theorem, every inverse category has an associated groupoid $\mathcal{G}(\mathbf{X})$:

Groupoids Associated to Inverse Categories

As a generalization of the Ehresmann-Schein-Nambooripad Theorem, every inverse category has an associated groupoid $\mathcal{G}(\mathbf{X})$:

- Objects: All of the $\overline{f}$'s for each $f : A \to B$ in $\mathbf{X}$.

Groupoids Associated to Inverse Categories

As a generalization of the Ehresmann-Schein-Nambooripad Theorem, every inverse category has an associated groupoid $\mathcal{G}(\mathbf{X})$:

- Objects: All of the $\overline{f}$'s for each $f : A \to B$ in $\mathbf{X}$.
- Arrows: For each arrow $f : A \to B$ in $\mathbf{X}$, an arrow $f : \overline{f} \to \overline{f^\circ}$.

Groupoids Associated to Inverse Categories

As a generalization of the Ehresmann-Schein-Nambooripad Theorem, every inverse category has an associated groupoid $\mathcal{G}(\mathbf{X})$:

- Objects: All of the $\overline{f}$'s for each $f : A \to B$ in $\mathbf{X}$.
- Arrows: For each arrow $f : A \to B$ in $\mathbf{X}$, an arrow $f : \overline{f} \to \overline{f^\circ}$.
  - Composition: for arrows $f : \overline{f} \to \overline{f^\circ}$ and $g : \overline{g} \to \overline{g^\circ}$ with $\overline{f^\circ} = \overline{g}$, we define their composite $g \bullet f : \overline{f} \to \overline{g^\circ}$ in $\mathcal{G}(\mathbf{X})$ to be their composite in $\mathbf{X}$.

Groupoids Associated to Inverse Categories

As a generalization of the Ehresmann-Schein-Nambooripad Theorem, every inverse category has an associated groupoid $\mathcal{G}(\mathbf{X})$:

- Objects: All of the $\overline{f}$'s for each $f : A \to B$ in $\mathbf{X}$.
- Arrows: For each arrow $f : A \to B$ in $\mathbf{X}$, an arrow $f : \overline{f} \to \overline{f^\circ}$.
  - Composition: for arrows $f : \overline{f} \to \overline{f^\circ}$ and $g : \overline{g} \to \overline{g^\circ}$ with $\overline{f^\circ} = \overline{g}$, we define their composite $g \bullet f : \overline{f} \to \overline{g^\circ}$ in $\mathcal{G}(\mathbf{X})$ to be their composite in $\mathbf{X}$.
  - Identities: For any object $\overline{f} : A \to A$ in $\mathcal{G}(\mathbf{X})$, define $1_{\overline{f}} = \overline{f}$.

Groupoids Associated to Inverse Categories

As a generalization of the Ehresmann-Schein-Nambooripad Theorem, every inverse category has an associated groupoid $\mathcal{G}(\mathbf{X})$:

- Objects: All of the $\overline{f}$'s for each $f : A \to B$ in $\mathbf{X}$.
- Arrows: For each arrow $f : A \to B$ in $\mathbf{X}$, an arrow $f : \overline{f} \to \overline{f^\circ}$.
  - Composition: for arrows $f : \overline{f} \to \overline{f^\circ}$ and $g : \overline{g} \to \overline{g^\circ}$ with $\overline{f^\circ} = \overline{g}$, we define their composite $g \bullet f : \overline{f} \to \overline{g^\circ}$ in $\mathcal{G}(\mathbf{X})$ to be their composite in $\mathbf{X}$.
  - Identities: For any object $\overline{f} : A \to A$ in $\mathcal{G}(\mathbf{X})$, define $1_{\overline{f}} = \overline{f}$.
  - Inverses: Given an arrow $f : \overline{f} \to \overline{f^\circ}$, define $f^{-1} : \overline{f^\circ} \to \overline{f}$ to be f°, the unique restricted inverse of f from $\mathbf{X}$'s inverse structure.

Groupoids Associated to Inverse Categories

As a generalization of the Ehresmann-Schein-Nambooripad Theorem, every inverse category has an associated groupoid $\mathcal{G}(\mathbf{X})$:

- Objects: All of the $\overline{f}$'s for each $f : A \to B$ in $\mathbf{X}$.
- Arrows: For each arrow $f : A \to B$ in $\mathbf{X}$, an arrow $f : \overline{f} \to \overline{f^\circ}$.
  - Composition: for arrows $f : \overline{f} \to \overline{f^\circ}$ and $g : \overline{g} \to \overline{g^\circ}$ with $\overline{f^\circ} = \overline{g}$, we define their composite $g \bullet f : \overline{f} \to \overline{g^\circ}$ in $\mathcal{G}(\mathbf{X})$ to be their composite in $\mathbf{X}$.
  - Identities: For any object $\overline{f} : A \to A$ in $\mathcal{G}(\mathbf{X})$, define $1_{\overline{f}} = \overline{f}$.
  - Inverses: Given an arrow $f : \overline{f} \to \overline{f^\circ}$, define $f^{-1} : \overline{f^\circ} \to \overline{f}$ to be f°, the unique restricted inverse of f from $\mathbf{X}$'s inverse structure.

Interesting Fact: This groupoid is what we call a *top-heavy locally inductive groupoid*: the objects can be partitioned into meet-semilattices and every arrow can be restricted to smaller source objects (and corestricted to smaller targets).

More Interesting Facts

More Interesting Facts

Fact: The partition of the objects into meet-semilattices is done by "anchoring" $\overline{f}$'s to their source objects:

For each object A in $\mathbf{X}$, the set

$$E_A = \left\{ \overline{f} : A \to A | f : A \to B \in \mathbf{X} \right\}$$

More Interesting Facts

Fact: The partition of the objects into meet-semilattices is done by "anchoring" $\overline{f}$'s to their source objects:

For each object A in $\mathbf{X}$, the set

$$E_A = \left\{\overline{f} : A \to A | f : A \to B \in \mathbf{X}\right\}$$

is a meet-semilattice:

- $f \leq g$ iff $g\overline{f} = f$
- $\overline{f} \wedge \overline{g} = \overline{f}\,\overline{g}$
- Each E_A has a top element $\top = \mathrm{id}_A$ (top-heavy)
- These E_A partition $\mathcal{G}(\mathbf{X})_0$ (locally inductive)

More Interesting Facts

More Interesting Facts

Fact: The functor $\mathcal{G}$ is part of an equivalence of categories between the categories of top-heavy locally inductive groupoids and of inverse categories.

More Interesting Facts

Fact: The functor $\mathcal{G}$ is part of an equivalence of categories between the categories of top-heavy locally inductive groupoids and of inverse categories.

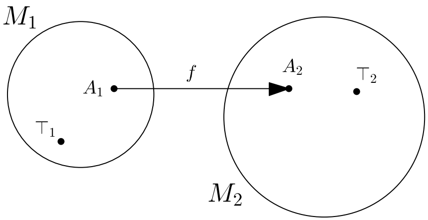
Given a top-heavy locally inductive groupoid $\mathbf{G}$, we can construct an inverse category $\mathcal{I}(\mathbf{G})$:

More Interesting Facts

Fact: The functor $\mathcal{G}$ is part of an equivalence of categories between the categories of top-heavy locally inductive groupoids and of inverse categories.

Given a top-heavy locally inductive groupoid **G**, we can construct an inverse category $\mathcal{I}(\mathbf{G})$:

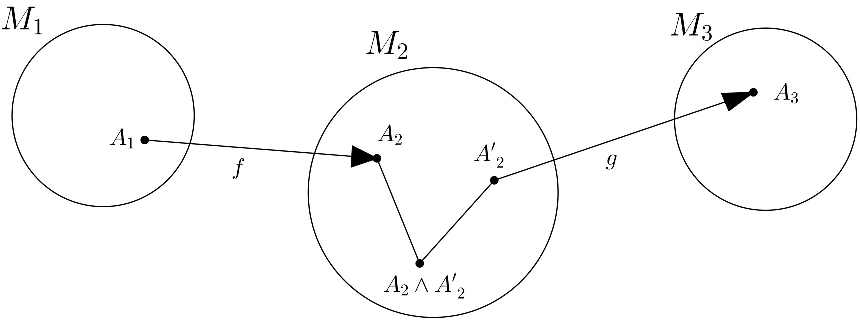
The objects of the inverse category are the meet-semilattices M_i and an arrow $M_i \rightarrow M_j$ exists for each $f : A_i \rightarrow A_j$ in **G** with $A_i \in M_i$:



More Interesting Facts

More Interesting Facts

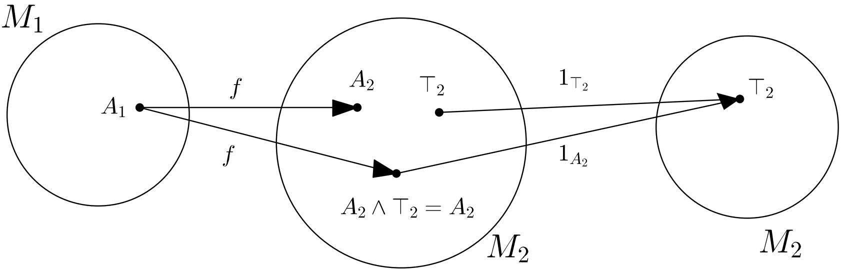
Composition in $\mathcal{I}(\mathbf{G})$ is defined using the restriction and corestriction of the top-heavy locally localic groupoid:



This composite is called $g \otimes f$.
Remarkably strictly associative.

More Interesting Facts

Also has identities:



This is, in fact, why the groupoids must be top-heavy for the equivalence to work.

Glueing Arrows

The ability to glue two maps together will come in handy later in defining topologies on the groupoids associated to inverse semigroups.

Glueing Arrows

The ability to glue two maps together will come in handy later in defining topologies on the groupoids associated to inverse semigroups.

Two maps will be glueable if they are *compatible* in the usual sense: that they (and their inverses) agree everywhere that they are both defined.

Glueing Arrows

The ability to glue two maps together will come in handy later in defining topologies on the groupoids associated to inverse semigroups.

Two maps will be glueable if they are *compatible* in the usual sense: that they (and their inverses) agree everywhere that they are both defined.

Definition

Let $\mathbf{X}$ be an inverse category. Two arrows f and g in $\mathbf{X}$ are *compatible* – denoted $f \smile g$ – if and only if $f\overline{g} = g\overline{f}$ and $f^\circ\overline{g^\circ} = g^\circ\overline{f^\circ}$.
A subset $S \subseteq \mathbf{X}_1$ of arrows in $\mathbf{X}$ is called a *compatible set* whenever every pair of arrows in S is compatible.

Glueing Arrows

The ability to glue two maps together will come in handy later in defining topologies on the groupoids associated to inverse semigroups.

Two maps will be glueable if they are *compatible* in the usual sense: that they (and their inverses) agree everywhere that they are both defined.

Definition

Let $\mathbf{X}$ be an inverse category. Two arrows f and g in $\mathbf{X}$ are *compatible* – denoted $f \smile g$ – if and only if $f\overline{g} = g\overline{f}$ and $f^\circ\overline{g^\circ} = g^\circ\overline{f^\circ}$.
A subset $S \subseteq \mathbf{X}_1$ of arrows in $\mathbf{X}$ is called a *compatible set* whenever every pair of arrows in S is compatible.

Coming up: some interesting compatible sets. But first, joins!

Adding Joins to our Inverse Categories

Definition (Cockett/Cruttwell/Gallagher, 2011)

A *join inverse category* is an inverse category in which for every compatible set $(f_i : A \to B)_{i \in I}$, there is a map $\bigvee_{i \in I} f_i : A \to B$ such that

Adding Joins to our Inverse Categories

Definition (Cockett/Cruttwell/Gallagher, 2011)

A *join inverse category* is an inverse category in which for every compatible set $(f_i : A \to B)_{i \in I}$, there is a map $\bigvee_{i \in I} f_i : A \to B$ such that

(i) for all $i \in I$, $f_i \leq \bigvee_{i \in I} f_i$,

Adding Joins to our Inverse Categories

Definition (Cockett/Cruttwell/Gallagher, 2011)

A *join inverse category* is an inverse category in which for every compatible set $(f_i : A \to B)_{i \in I}$, there is a map $\bigvee_{i \in I} f_i : A \to B$ such that

(i) for all $i \in I$, $f_i \leq \bigvee_{i \in I} f_i$,

(ii) if there exists a map g such that $f_i \leq g$ for all $i \in I$, then $\bigvee f_i \leq g$,

Adding Joins to our Inverse Categories

Definition (Cockett/Cruttwell/Gallagher, 2011)

A *join inverse category* is an inverse category in which for every compatible set $(f_i : A \to B)_{i \in I}$, there is a map $\bigvee_{i \in I} f_i : A \to B$ such that

(i) for all $i \in I$, $f_i \leq \bigvee_{i \in I} f_i$,

(ii) if there exists a map g such that $f_i \leq g$ for all $i \in I$, then $\bigvee f_i \leq g$,

(iii) for any $h : B \to C$, $h\left(\bigvee_{i \in I} f_i\right) = \bigvee_{i \in I} hf_i$.

Adding Joins to our Inverse Categories

Definition (Cockett/Cruttwell/Gallagher, 2011)

A *join inverse category* is an inverse category in which for every compatible set $(f_i : A \to B)_{i \in I}$, there is a map $\bigvee_{i \in I} f_i : A \to B$ such that

(i) for all $i \in I$, $f_i \leq \bigvee_{i \in I} f_i$,

(ii) if there exists a map g such that $f_i \leq g$ for all $i \in I$, then $\bigvee f_i \leq g$,

(iii) for any $h : B \to C$, $h \left(\bigvee_{i \in I} f_i \right) = \bigvee_{i \in I} hf_i$.

Facts:

(i) for any $j \in I$, $\left(\bigvee_{i \in I} f_i \right) \overline{f_j} = f_j$,

(ii) for any $h : C \to A$, $\left(\bigvee_{i \in I} f_i \right) h = \bigvee_{i \in I} f_i h$,

(iii) $\overline{\bigvee_{i \in I} f_i} = \bigvee_{i \in I} \overline{f_i}$.

Principal Order Ideals

Definition

For each object $\overline{f}$ in $\mathcal{G}(\mathbf{X})$, the *principal order ideal* of $\overline{f}$ is the set of objects

$$\downarrow \overline{f} = \left\{ \overline{e} \in \mathcal{G}(\mathbf{X})_0 : \overline{e} \leq \overline{f} \right\}.$$

Principal Order Ideals

Definition

For each object $\overline{f}$ in $\mathcal{G}(\mathbf{X})$, the *principal order ideal* of $\overline{f}$ is the set of objects

$$\downarrow \overline{f} = \left\{ \overline{e} \in \mathcal{G}(\mathbf{X})_0 : \ \overline{e} \leq \overline{f} \right\}.$$

Proposition

For each object $\overline{f} \in \mathcal{G}(\mathbf{X})$, the principal order ideal $\downarrow \overline{f}$ is a compatible set.

Principal Order Ideals

Definition

For each object $\overline{f}$ in $\mathcal{G}(\mathbf{X})$, the *principal order ideal* of $\overline{f}$ is the set of objects

$$\downarrow \overline{f} = \left\{ \overline{e} \in \mathcal{G}(\mathbf{X})_0 : \ \overline{e} \leq \overline{f} \right\}.$$

Proposition

For each object $\overline{f} \in \mathcal{G}(\mathbf{X})$, the principal order ideal $\downarrow \overline{f}$ is a compatible set.

Proposition

Let $\mathbf{X}$ be a join inverse category. For each object $\overline{f} \in \mathcal{G}(\mathbf{X})$, the principal order ideal $\downarrow \overline{f}$ is a locale with all joins inherited from $\mathbf{X}$ and meet defined by $\overline{a} \wedge \overline{b} = \overline{a}\overline{b}$.

Locale-Valued Functor

Proposition

Let **X** *be a join inverse category. For each arrow* $\alpha : \overline{\alpha} \to \overline{\alpha^\circ}$ *in* $\mathcal{G}(\mathbf{X})$, *there is a frame homomorphism* $\alpha_* :\downarrow \overline{\alpha} \to \downarrow \overline{\alpha^\circ}$ *defined by* $\alpha_* \left(\overline{b}\right) = \overline{\overline{b}\,\alpha^\circ}$.

Locale-Valued Functor

Proposition

Let $\mathbf{X}$ *be a join inverse category. For each arrow* $\alpha : \overline{\alpha} \to \overline{\alpha^\circ}$ *in* $\mathcal{G}(\mathbf{X})$, *there is a frame homomorphism* $\alpha_* : \downarrow \overline{\alpha} \to \downarrow \overline{\alpha^\circ}$ *defined by* $\alpha_*\left(\overline{b}\right) = \overline{\overline{b}\,\alpha^\circ}$.

$$A \underset{\alpha^\circ}{\overset{\alpha}{\rightleftarrows}} B$$

Locale-Valued Functor

Proposition

Let **X** *be a join inverse category. For each arrow* $\alpha : \overline{\alpha} \to \overline{\alpha^\circ}$ *in* $\mathcal{G}(\mathbf{X})$, *there is a frame homomorphism* $\alpha_* : \downarrow \overline{\alpha} \to \downarrow \overline{\alpha^\circ}$ *defined by* $\alpha_*\left(\overline{b}\right) = \overline{\overline{b}\,\alpha^\circ}$.

$$A \underset{\alpha^\circ}{\overset{\alpha}{\rightleftarrows}} B \;\Rightarrow\; \begin{array}{c} \overline{b} \in \downarrow \overline{\alpha} \\ A \underset{\overline{b}}{\overset{\overline{\alpha}}{\rightrightarrows}} A \end{array}$$

Locale-Valued Functor

Proposition

Let $\mathbf{X}$ *be a join inverse category. For each arrow* $\alpha : \overline{\alpha} \to \overline{\alpha^\circ}$ *in* $\mathcal{G}(\mathbf{X})$, *there is a frame homomorphism* $\alpha_* :\downarrow \overline{\alpha} \to \downarrow \overline{\alpha^\circ}$ *defined by* $\alpha_*\left(\overline{b}\right) = \overline{\overline{b}\,\alpha^\circ}$.

$$A \underset{\alpha^\circ}{\overset{\alpha}{\rightleftarrows}} B \;\Rightarrow\; \begin{array}{c} \overline{b} \in \downarrow \overline{\alpha} \\ A \underset{\overline{b}}{\overset{\overline{\alpha}}{\rightrightarrows}} A \end{array} \;\Rightarrow\; B \xrightarrow{\alpha^\circ} A \xrightarrow{\overline{b}} A$$

Locale-Valued Functor

Proposition

Let $\mathbf{X}$ *be a join inverse category. For each arrow* $\alpha : \overline{\alpha} \to \overline{\alpha^\circ}$ *in* $\mathcal{G}(\mathbf{X})$, *there is a frame homomorphism* $\alpha_* : \downarrow \overline{\alpha} \to \downarrow \overline{\alpha^\circ}$ *defined by* $\alpha_*\left(\overline{b}\right) = \overline{\overline{b}\,\alpha^\circ}$.

$$\overline{b} \in \downarrow \overline{\alpha}$$

$$A \underset{\alpha^\circ}{\overset{\alpha}{\rightleftarrows}} B \;\Rightarrow\; A \underset{\overline{b}}{\overset{\overline{\alpha}}{\rightrightarrows}} A \;\Rightarrow\; B \xrightarrow{\alpha^\circ} A \xrightarrow{\overline{b}} A \;\Rightarrow\; B \xrightarrow{\overline{\overline{b}\,\alpha^\circ}} B$$

Locale-Valued Functor

Proposition

Let $\mathbf{X}$ *be a join inverse category. For each arrow* $\alpha : \overline{\alpha} \to \overline{\alpha^\circ}$ *in* $\mathcal{G}(\mathbf{X})$, *there is a frame homomorphism* $\alpha_* :\downarrow \overline{\alpha} \to \downarrow \overline{\alpha^\circ}$ *defined by* $\alpha_*\left(\overline{b}\right) = \overline{\overline{b}\,\alpha^\circ}$.

$$A \underset{\alpha^\circ}{\overset{\alpha}{\rightleftarrows}} B \;\Rightarrow\; \overline{b} \in \downarrow \overline{\alpha},\quad A \underset{\overline{b}}{\overset{\overline{\alpha}}{\rightrightarrows}} A \;\Rightarrow\; B \xrightarrow{\alpha^\circ} A \xrightarrow{\overline{b}} A \;\Rightarrow\; B \xrightarrow{\overline{\overline{b}\,\alpha^\circ}} B$$

Corollary

Let $\mathbf{X}$ *be a join inverse category. There is a contravariant functor*

$$(-)_* : \mathcal{G}(\mathbf{X})^{\mathrm{op}} \to \mathbf{Loc},$$

where **Loc** *is the category of locales and locale morphisms.*

Locale-Valued Functor, redux

Lemma

Let $\mathbf{X}$ *be a join inverse category. For each arrow* $\alpha : \overline{\alpha} \to \overline{\alpha^\circ}$ *in* $\mathcal{G}(\mathbf{X})$, *there is a frame homomorphism* $\alpha^* : \downarrow \overline{\alpha^\circ} \to \downarrow \overline{\alpha}$ *defined by* $\alpha^*(\overline{e}) = \overline{\overline{e}\,\alpha}$. □

Locale-Valued Functor, redux

Lemma

Let $\mathbf{X}$ *be a join inverse category. For each arrow* $\alpha : \overline{\alpha} \to \overline{\alpha^\circ}$ *in* $\mathcal{G}(\mathbf{X})$*, there is a frame homomorphism* $\alpha^* : \downarrow \overline{\alpha^\circ} \to \downarrow \overline{\alpha}$ *defined by* $\alpha^*(\overline{e}) = \overline{\overline{e}\,\alpha}$. □

Corollary

Let $\mathbf{X}$ *be a join inverse category. There is a covariant functor*

$$(-)^* : \mathcal{G}(\mathbf{X}) \to \mathbf{Loc},$$

where **Loc** *is the category of locales and locale morphisms.* □

Fact Number 1 from Abstract

Theorem

Let **X** *be a join inverse category. For each arrow* $\alpha : \overline{\alpha} \to \overline{\alpha^\circ}$ *in* $\mathcal{G}(\mathbf{X})$, *there is an equivalence of (locales as) categories*

$$\downarrow \overline{\alpha} \underset{\alpha^*}{\overset{\alpha_*}{\rightleftarrows}} \downarrow \overline{\alpha^\circ} \qquad \square$$

Fact Number 1 from Abstract

Theorem

Let $\mathbf{X}$ *be a join inverse category. For each arrow* $\alpha : \overline{\alpha} \to \overline{\alpha^\circ}$ *in* $\mathcal{G}(\mathbf{X})$*, there is an equivalence of (locales as) categories*

$$\downarrow \overline{\alpha} \underset{\alpha^*}{\overset{\alpha_*}{\rightleftarrows}} \downarrow \overline{\alpha^\circ} \qquad \square$$

Local topological data seems to suggest some Grothendieck-topology-styled structure could be used to organize this information.

Fact Number 1 from Abstract

Theorem

Let $\mathbf{X}$ *be a join inverse category. For each arrow* $\alpha : \overline{\alpha} \to \overline{\alpha^\circ}$ *in* $\mathcal{G}(\mathbf{X})$, *there is an equivalence of (locales as) categories*

$$\downarrow \overline{\alpha} \underset{\alpha^*}{\overset{\alpha_*}{\rightleftarrows}} \downarrow \overline{\alpha^\circ} \qquad \square$$

Local topological data seems to suggest some Grothendieck-topology-styled structure could be used to organize this information.

The top-heavy-locally inductive groupoid associated to an inverse category is an example of an *ordered groupoid*.

Fact Number 1 from Abstract

Theorem

Let $\mathbf{X}$ *be a join inverse category. For each arrow* $\alpha : \overline{\alpha} \to \overline{\alpha^\circ}$ *in* $\mathcal{G}(\mathbf{X})$, *there is an equivalence of (locales as) categories*

$$\downarrow \overline{\alpha} \underset{\alpha^*}{\overset{\alpha_*}{\rightleftarrows}} \downarrow \overline{\alpha^\circ} \qquad \square$$

Local topological data seems to suggest some Grothendieck-topology-styled structure could be used to organize this information.

The top-heavy-locally inductive groupoid associated to an inverse category is an example of an *ordered groupoid*.

Mark Lawson and Benjamin Steinberg have explored topological structures on ordered groupoids. Their work motivates Fact Number 2 from Abstract.

Quick Detour: Another Partial Order

Definition (Lawson, 2004)

Define a relation $\leq_{\mathcal{J}}$ on the objects of a top-heavy locally inductive groupoid by $a \leq_{\mathcal{J}} b$ if and only if there exists an object $a' \cong a$ such that $a' \leq b$.

That is, a is isomorphic to some object sitting below b :

$$\begin{array}{ccc} & & B \\ & & \rotatebox{90}{$\geq$} \\ A & \xrightarrow{f} & B' \end{array}$$

Quick Detour: Another Partial Order

Definition (Lawson, 2004)

Define a relation $\leq_{\mathcal{J}}$ on the objects of a top-heavy locally inductive groupoid by $a \leq_{\mathcal{J}} b$ if and only if there exists an object $a' \cong a$ such that $a' \leq b$.

That is, a is isomorphic to some object sitting below b :

$$\begin{array}{ccc} & & B \\ & & \vee\!| \\ A & \xrightarrow{f} & B' \end{array}$$

NB Two $\leq_{\mathcal{J}}$-related objects permit composition using $\otimes$:

$$\begin{array}{ccccc} & & B & \xrightarrow{g} & C \\ & & \vee\!| & & \\ A & \xrightarrow{f} & B' & & \end{array} \quad \longmapsto \quad A \xrightarrow{f} B' \xrightarrow{g|_{B'}} C$$

Ehresmann Topologies

Definition (Lawson/Steinberg, 2004)

Let $(\mathbf{G}, \circ, \leq)$ be an ordered groupoid. An *Ehresmann topology* on $\mathbf{G}$ is an assignment of, for each object $e \in \mathbf{G}$, a collection $T(e)$ of order ideals of $\downarrow e$ – called *covering ideals* – satisfying

Ehresmann Topologies

Definition (Lawson/Steinberg, 2004)

Let $(\mathbf{G}, \circ, \leq)$ be an ordered groupoid. An *Ehresmann topology* on $\mathbf{G}$ is an assignment of, for each object $e \in \mathbf{G}$, a collection $T(e)$ of order ideals of $\downarrow e$ – called *covering ideals* – satisfying

(i) $\downarrow e \in T(e)$ for each object $e \in \mathbf{G}$.

(ii) Let e and f be objects of $\mathbf{G}$ such that $f \leq_{\mathcal{J}} e$. Then for each $x : f \cong e' \leq e$ and $\mathcal{A} \in T(e)$, we have $x^{-1} \otimes \mathcal{A} \otimes x \in T(f)$.

(iii) Let e be an object of $\mathbf{G}$, let $\mathcal{A} \in T(e)$ and let $\mathcal{B} \trianglelefteq \downarrow e$ be an arbitrary order ideal of $\downarrow e$. Suppose that, for each $x : f \cong e' \leq e$ (where $e' \in \mathcal{A}$), we have $x^{-1} \otimes \mathcal{B} \otimes x \in T(f)$. Then $\mathcal{B} \in T(e)$.

An ordered groupoid equipped with an Ehresmann topology is an *Ehresmann site*.

Fact Number 2 from Abstract

Theorem

If $\mathbf{X}$ *is a join inverse category, then* $\mathcal{G}(\mathbf{X})$ *admits an Ehresmann site with, for each object* $\overline{e} \in \mathcal{G}(\mathbf{X})$,

$$T(\overline{e}) = \left\{ \mathcal{S} \trianglelefteq \downarrow \overline{e} : \bigvee \mathcal{S} = \overline{e} \right\}.$$

Fact Number 2 from Abstract

Theorem

If $\mathbf{X}$ *is a join inverse category, then* $\mathcal{G}(\mathbf{X})$ *admits an Ehresmann site with, for each object* $\overline{e} \in \mathcal{G}(\mathbf{X})$,

$$T(\overline{e}) = \left\{ \mathcal{S} \trianglelefteq \downarrow \overline{e} : \bigvee \mathcal{S} = \overline{e} \right\}.$$

Interesting Facts:

- Lawson/Steinberg (2004): to each Ehresmann site can be associated a left-cancellative site.

Fact Number 2 from Abstract

Theorem

If $\mathbf{X}$ *is a join inverse category, then* $\mathcal{G}(\mathbf{X})$ *admits an Ehresmann site with, for each object* $\overline{e} \in \mathcal{G}(\mathbf{X})$,

$$T(\overline{e}) = \left\{ \mathcal{S} \trianglelefteq \downarrow \overline{e} : \bigvee \mathcal{S} = \overline{e} \right\}.$$

Interesting Facts:

- Lawson/Steinberg (2004): to each Ehresmann site can be associated a left-cancellative site.
- Lawson/Steinberg (2004) via Kock/Moerdijk (1991): every étendue is equivalent to a topos of sheaves on an Ehresmann site.

Fact Number 2 from Abstract

Theorem

If $\mathbf{X}$ *is a join inverse category, then* $\mathcal{G}(\mathbf{X})$ *admits an Ehresmann site with, for each object* $\overline{e} \in \mathcal{G}(\mathbf{X})$*,*

$$T(\overline{e}) = \left\{ \mathcal{S} \trianglelefteq \downarrow \overline{e} : \bigvee \mathcal{S} = \overline{e} \right\}.$$

Interesting Facts:

- Lawson/Steinberg (2004): to each Ehresmann site can be associated a left-cancellative site.
- Lawson/Steinberg (2004) via Kock/Moerdijk (1991): every étendue is equivalent to a topos of sheaves on an Ehresmann site.

Question: What class of étendues is obtained by restricting this construction to the Ehresmann sites coming from inverse categories?