

CT 2018Preamble

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Quillen localization (in the sense of Gabriel-Zisman) can be analyzed as a 2-step construction.

FIRST Localize at \mathcal{W} = weak equivalences
the subcategory of fibrant-cofibrant objects

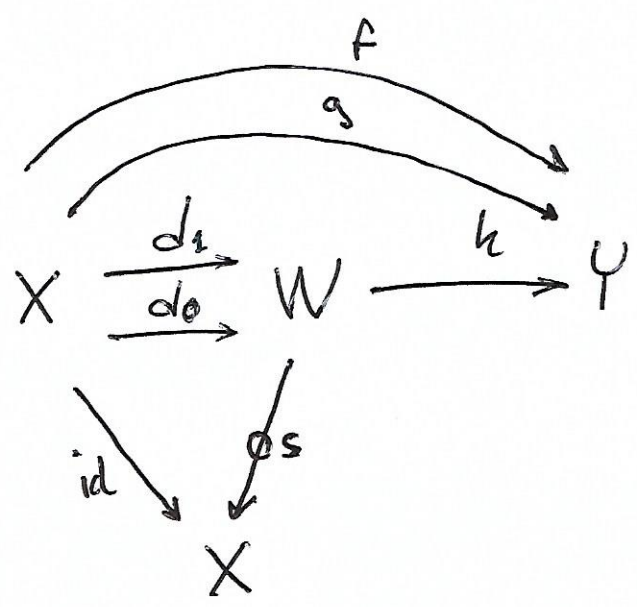
SECOND Construct the fibrant-cofibrant replacement

In joint work with Szlyd and Descotte we developed a 2-dimensional version of these constructions.

Localizations are taken in the sense of Quillen.

I will talk only on the FIRST step, and furthermore in the case in which the starting data is a model category in the usual sense.

Question

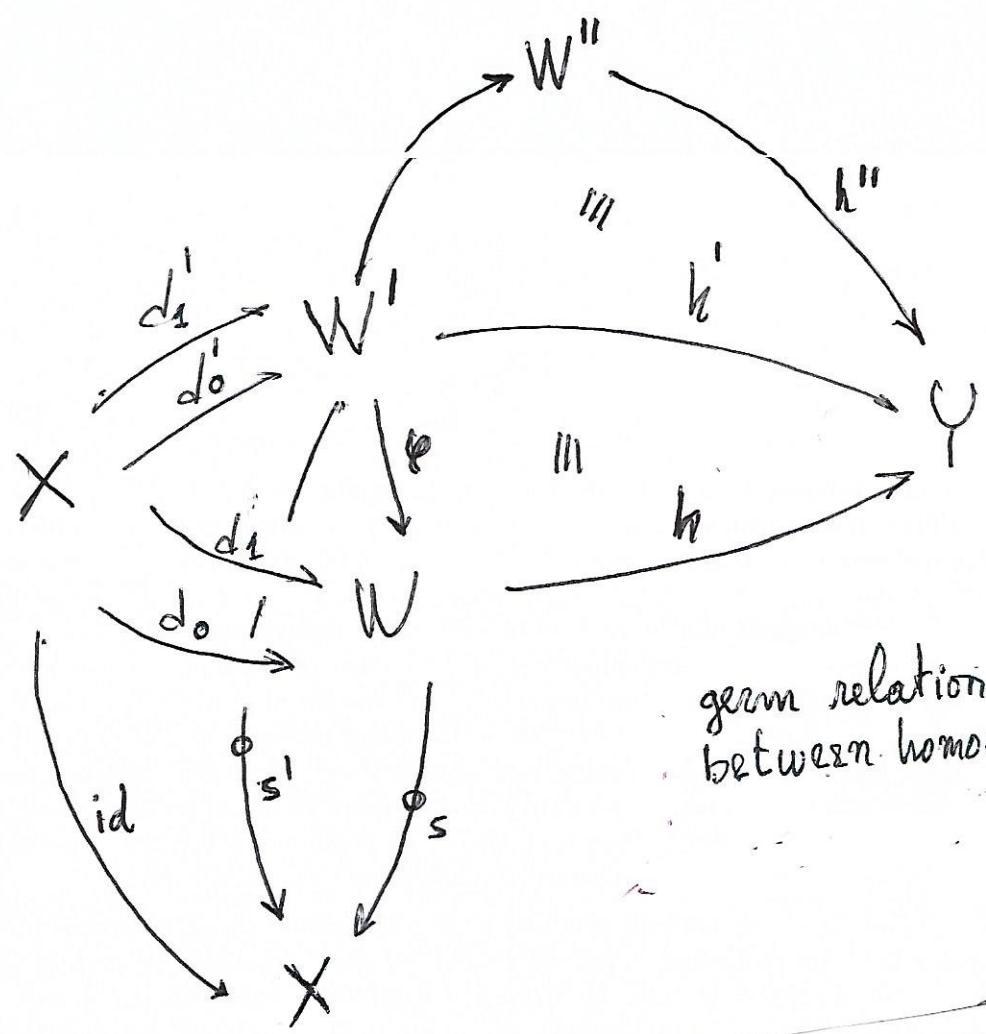


Cylinder C

Homotopy $H = (C, h)$

$$f \xrightarrow{H} g$$

Morphism of cylinder determines equivalence relation



germ relation between homotopies

Cylinders $(X)^{op} \xrightarrow{hpy_{fg}(-, Y)} \text{Set}$

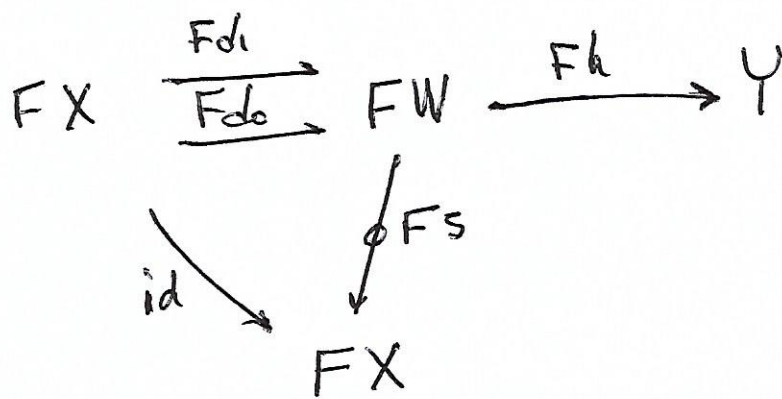
colimit

$$\{ H = (C, h) : f \rightsquigarrow g \}$$

Given $\mathcal{C} \xrightarrow{F} \mathcal{D}$ 2 Functor and $H: \mathcal{F} \rightsquigarrow \mathcal{G}$

$H = (C, h)$. Then $FH = (FC, Fh): \mathcal{F}\mathcal{F} \rightsquigarrow \mathcal{F}\mathcal{G}$

Now F_s is a real equivalence:



get $\widehat{FH} = F_h \widehat{FC} : \mathcal{F}\mathcal{F} \Rightarrow \mathcal{F}\mathcal{G}$

where $\widehat{FC} : F_{d1} \Rightarrow F_{d0}$ UNIQUE | $F_s \widehat{FC} = \text{id}$

Define $F[H] \stackrel{\text{def}}{=} \widehat{FH}$

where $[H]$ is equivalence class of " \rightsquigarrow " germ relation

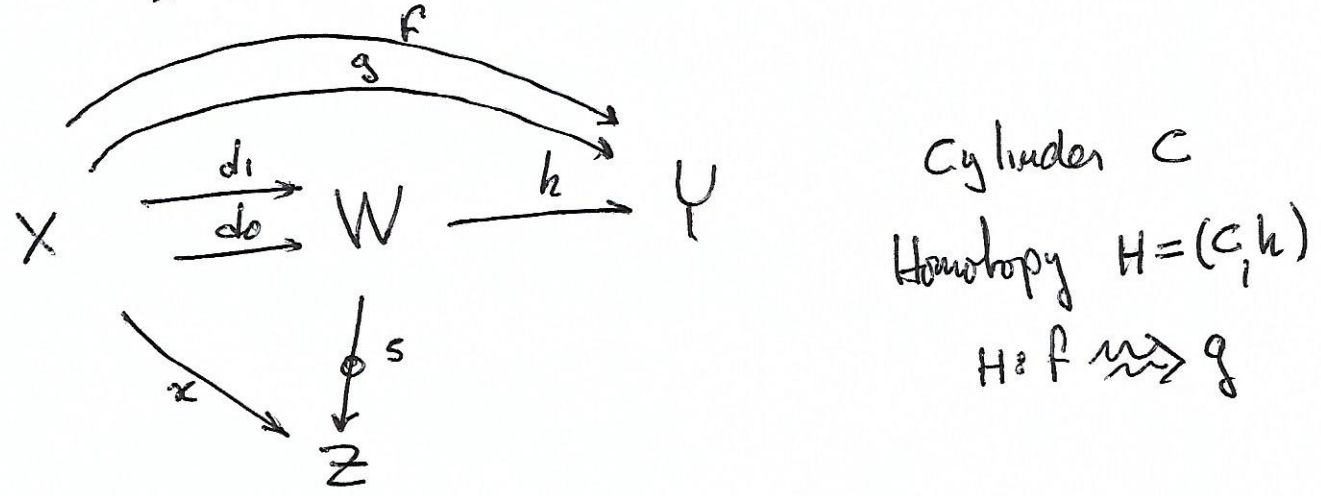
Easy to prove $H \rightsquigarrow H' \Rightarrow \widehat{FH} = \widehat{FH}'$

so well defined .

But still not have a 2-category .

At this point Szjld's idea :

- a) Considers a generalization of Quillen's cylinders.
- b) Forget Model Categories and considers only a single class Σ (containing the identities) in a category \mathcal{A}



As before

s equivalence \Rightarrow $\boxed{\exists! \hat{C} : d_1 \Rightarrow d_0 \mid s\hat{C} = \kappa}$
 $\hat{H} = h\hat{C} : \underset{\hat{h}d_1}{f} \Rightarrow \underset{\hat{h}d_0}{g}$

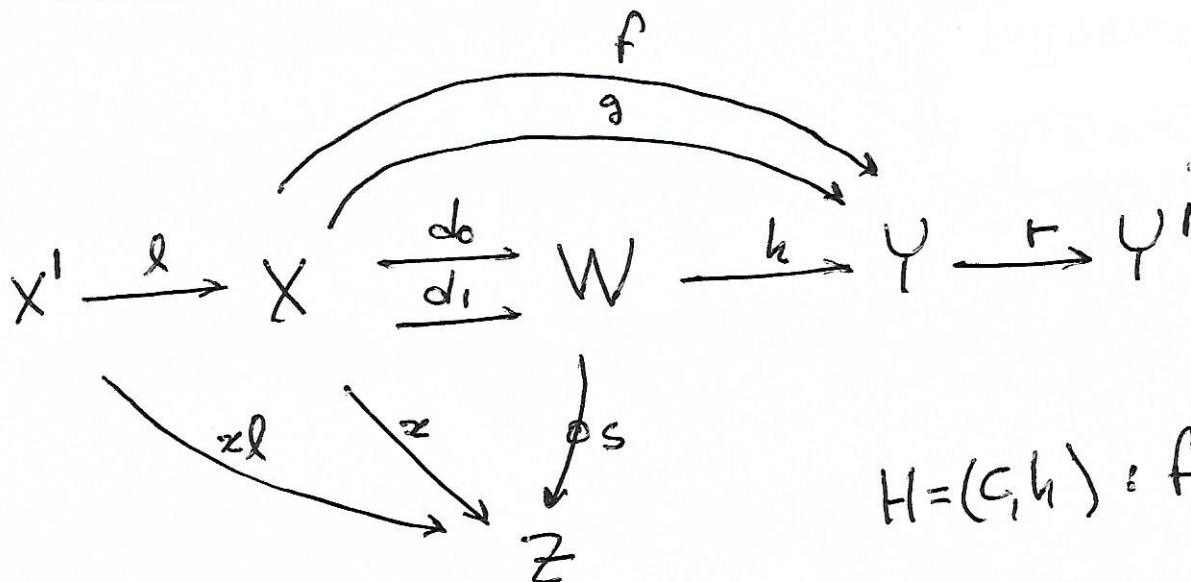
given $\mathcal{A} \xrightarrow{F} \mathcal{D}$ define $\boxed{F[H] : Ff \Rightarrow fg \text{ in } \mathcal{D}}$
 $F[H] = \hat{F}H$

With this we decided Forget germ relation (for the moment)

Define Ad Hoc relation $\boxed{H \sim H' \Leftrightarrow \hat{F}H = \hat{F}H' \quad \forall F}$

Things become simpler.

Horizontal Composition $\xrightarrow{l} \xrightarrow{H} \xrightarrow{r}$



$$H = (c, h) : f \rightsquigarrow g$$

$$Hl = (cl, h) : fl \rightsquigarrow gl \quad (\text{same } h)$$

$$rH = (c, rh) : rf \rightsquigarrow rg \quad (\text{same } c)$$

Example of simple reasoning with the AdHoc relation

$$\widehat{Hl} = \widehat{H}l$$

$$\widehat{cl} = \widehat{c}l : s(\widehat{c}l) = (s\widehat{c})l = xl$$

$$\text{then } \widehat{H}l = h\widehat{cl} = h\widehat{c}l = \widehat{H}l$$

Define $\widehat{[H]l} = \widehat{[Hl]}$. Well defined :

$$\widehat{F(Hl)} = \widehat{FHFl} = \widehat{FH}Fl = \widehat{FH'}Fl = \widehat{FH'Fl} = \widehat{F(H'l)}$$

Composition with r even easier because H and rH have same cylinder.

Vertical Composition

Forget vertical composition of homotopies.

Take 2-cells as sequences of composable homotopies

$$f \underset{\sim}{\overset{H}{\rightrightarrows}} g \underset{\sim}{\overset{K}{\rightrightarrows}} t \qquad f \underset{\sim}{\overset{H'}{\rightrightarrows}} g' \underset{\sim}{\overset{K'}{\rightrightarrows}} t \qquad g \neq g'$$

$$(K, H) \sim (K', H') \iff \widehat{FK} \circ \widehat{FH} = \widehat{FK'} \circ \widehat{FH'}$$

in \mathcal{D}

Define $[K, H] \ell = [K\ell, H\ell]$ - well defined with no problem.

$r[K, H] = [rK, rH]$

Identities for horizontal composition same as \mathcal{A} .

Identities for vertical composition: $X \xrightarrow{f} Y$

$$\left| \begin{array}{ccc} X & \xrightarrow{id} & X & \xrightarrow{f} & Y \\ id \searrow & & \downarrow id & & \\ & & X & & \end{array} \right| \sim \left| \begin{array}{ccc} X & \xrightarrow{f} & Y & \xrightarrow{id} & Y \\ f \searrow & & \downarrow id & & \\ & & Y & & \end{array} \right|$$

Axioms of 2-category follow (apply F) from axioms of 2-category of \mathcal{D} and using equations

$$\widehat{Hl} = \widehat{H}l \quad \text{and} \quad r\widehat{H} = r\widehat{H} \quad .$$

Have $\mathcal{C} \xrightarrow{i} \widetilde{Ho}(\mathcal{C})$ $iX = X$
 $i f = f$

Universal Property

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{i} & \widetilde{Ho}(\mathcal{A}) \\
 \searrow F & \Downarrow \cong & \downarrow \exists! \widetilde{F} \\
 & & \mathcal{D}
 \end{array}$$

$F(\Sigma) \subset \text{Equivalences}$

$$\widetilde{F}X = FX \quad \widetilde{F}f = Ff$$

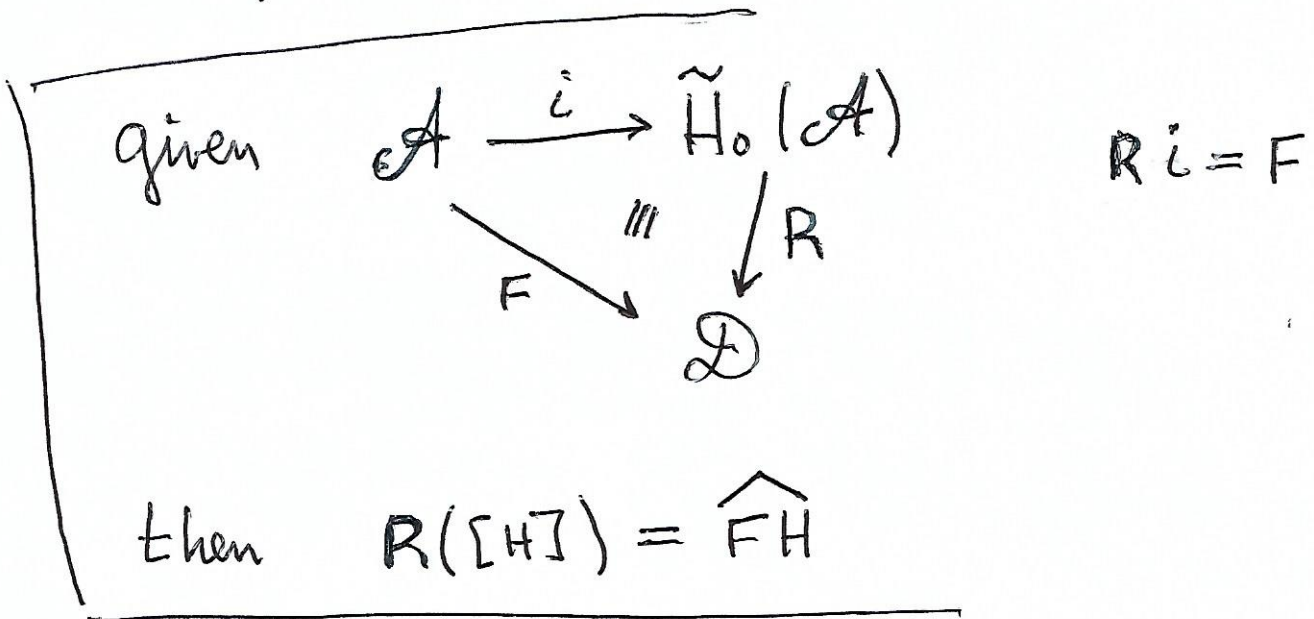
define

$$\begin{array}{l}
 \widetilde{F}([H]) = \widehat{FH} \\
 \widetilde{F}([H, K]) = \widehat{FH} \cdot \widehat{FK}
 \end{array}$$

By definition is 2-functor for vertical composition

Using equations $\widehat{Hl} = \widehat{H}l$ and $r\widehat{H} = r\widehat{H}$ it follows functoriality for horizontal composition.

Uniqueness :



Can also prove the 2-categorical aspect :

given $F \xrightarrow{\eta} G$ define $\tilde{F} \xrightarrow{\tilde{\eta}} \tilde{G}$

$Fx \xrightarrow{\eta_x} Gx$ $\tilde{F}x \xrightarrow{\tilde{\eta}_x} \tilde{G}x$

$\tilde{\eta}_x = \eta_x$

Conditions of 2-naturality for $\tilde{\eta}$ hold (not easy)

ONLY REMAINS TO PROVE THAT i SENDS THE CLASS Σ TO EQUIVALENCES

FACT All 2-cells of $\tilde{\text{Ho}}(\mathcal{A})$ are invertible

C , define C^{-1} by switching d_1 and d_0

$H = (c, h)$ define $H^{-1} = (C^{-1}, h)$

$f \xrightarrow{H} g$ clearly $g \xrightarrow{H^{-1}} f$

Can easily check $[H]^{-1} = [H^{-1}]$

It follows :

Σ

i) 3x2 property

(Loyal's terminology, pay 2 get 3)

ii) split generated

then $i \Sigma \subset \text{equivalences}$

Theorem

$\text{Hom}(\tilde{\text{Ho}}(\mathcal{A}), \mathcal{D}) \xrightarrow{i^*} \text{Hom}_+(\mathcal{A}, \mathcal{D})$

isomorphism of categories.

RECOVERING QUILLEN

Can show that germ relation \Rightarrow ad hoc relation

This is useful for some computations in what follows:

$\mathcal{A} = \mathcal{C}_{f,c} \subset \mathcal{C}$ \mathcal{C} a model category

$\Sigma = \mathcal{W}$ Conditions i), ii) hold.

Consider Quillen's cylinders.

Lemma

X, Y fibrant-cofibrant $X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} Y$

$H = (C, h) : f \rightsquigarrow g$

Then $\exists H' = (C', h') : f \rightsquigarrow g$

with C' a Quillen cylinder in \mathcal{C}_{cf}

and $H' \sim H$, $[H'] = [H]$

Lemma given $[H, K] \quad f \begin{array}{c} \xrightarrow{K} \\ \rightsquigarrow \end{array} g \begin{array}{c} \xrightarrow{H} \\ \rightsquigarrow \end{array} t$

$\exists f \begin{array}{c} \xrightarrow{N} \\ \rightsquigarrow \end{array} t \quad / \quad [N] = [H, K] \quad \text{IN PARTICULAR}$

TAKE π and obtain Quillen's result. 