CT 2018

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Preamble

Quillen Localization (in the sense of Gabriel-Zisman
can be analyzed as a 2-stept construction.

FIRST Localize at $\mathcal{W}$ = weak equivalences
the subcategory of fibrant-cofibrant objects

SECOND construct the fibrant-cofibrant replacement

In joint work with Szyld and Descotte we
developed a 2-dimensional version of these constructions

Localizations are taken in the sense of Pronk.

I will talk only on the FIRST stept,
and furthermore in the case in which the
starting data is a model category in the
usual sense.

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$\mathcal{C}$ model category $\quad \mathcal{D}$ a 2-category

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "i"] \arrow[dr, "\forall F"'] & \widetilde{Ho}(\mathcal{C}) \arrow[r] \arrow[d, "\exists"] & \pi_0 \widetilde{Ho}(\mathcal{C}) = Ho(\mathcal{C}) \arrow[dl, dashed, "\exists"] \\
& \mathcal{D} &
\end{tikzcd}$$

$F\mathcal{W} \subset$ Equivalences

Want $\quad$ Ob & Arr of $\widetilde{Ho}(\mathcal{C})$ same as $\mathcal{C}$

$\quad$ 2-cells of $\widetilde{Ho}(\mathcal{C})$ homotopies.

$f \sim g$ (homotopic) should be a 2-cell $f \Rightarrow g$.

Transitivity should become VERTICAL COMPOSITION

Congruency should become HORIZONTAL COMPOSITION

<u>NEED</u> equivalence relation between homotopies for axioms of a 2-category to hold.

Assume X Y fibrant cofibrant as needed.

Quillen

$$\begin{tikzcd}
X \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[rr, bend left=50, "f"] \arrow[rr, bend left=30, "g"'] \arrow[dr, "id"'] & W \arrow[r, "h"] \arrow[d, "s"] & Y \\
& X &
\end{tikzcd}$$

Cylinder C

Homotopy $H = (C, h$

$\boxed{f \overset{H}{\rightsquigarrow} g}$

Morphism of cylinder determines equivalence relation

$$\begin{tikzcd}
& & W'' \arrow[drr, "h''"] & & \\
& W' \arrow[ur] \arrow[rrr, "h'"] \arrow[d, "\varphi"] \arrow[ddl, "s'"'] & & & Y \\
X \arrow[ur, shift left, "d_1'"] \arrow[ur, shift right, "d_0'"'] \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[d, "id"'] & W \arrow[rrru, "h"] \arrow[dl, "s"] & & & \\
X & & & &
\end{tikzcd}$$

germ relation between homotopies

$$\mathrm{Cylinders}(X)^{op} \xrightarrow{\ hpy_{fg}(-, Y)\ } \mathrm{Set} \qquad \boxed{\text{colimit}}$$

$$C \longmapsto \{ H = (C, h) : f \rightsquigarrow g \}$$

Horizontal composition

$$\begin{tikzcd}
C' & C & \\
X'\sqcup X' \arrow[r, "\ell\sqcup\ell"] \arrow[d, "{\binom{d_1'}{d_0'}}"] \arrow[dd, bend right=60, "\triangledown"'] \arrow[dr, phantom, "\equiv"] & X\sqcup X \arrow[d, "{\binom{d_1}{d_0}}"] \arrow[dr, bend left, "{\binom{f}{g}}"] \arrow[dd, bend right=60, "\triangledown"'] \arrow[dr, phantom, "\equiv"'] & \\
W' \arrow[r, "\pi"] \arrow[d, "s'"] \arrow[dr, phantom, "pb"] & W \arrow[r, "h"] \arrow[d, "s"] & Y \\
X' \arrow[r, "\ell"] & X &
\end{tikzcd}$$

$(c, h) : f \rightsquigarrow g$ get $(c', h\pi) : f\ell \rightsquigarrow g\ell$

need s fibration.

Vertical composition.

$$\begin{tikzcd}
X \arrow[dr, "d_1"] & & & \\
& W \arrow[dr] \arrow[rr, bend left, "h"] & & \\
X \arrow[ur, "d_0"] \arrow[dr, "d_1'"'] & & W'' \arrow[r, dashed] \arrow[ddd, "s''"] & Y \\
& W' \arrow[ur] \arrow[urr, bend right, "h'"'] \arrow[ur, phantom, "p.o"] & & \\
X \arrow[ur, "d_0'"'] & & & \\
& & X &
\end{tikzcd}$$

$(c, h) : f \rightsquigarrow g$

$(c' h') : g \rightsquigarrow t$

get

$H' \circ H : f \rightsquigarrow t$

problem if require s'' fibration

WAY OUT ?

In the meantime:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "i"] \arrow[dr, "F"'] & \widetilde{Ho}(\mathcal{C}) \arrow[d, dashed, "F"] \\
& \mathcal{D}
\end{tikzcd}$$

How to define F in the homotopies

Consider $\mathcal{C}$ underline category of a 2-category with $\mathcal{W}$ the equivalences.

$$\begin{tikzcd}
X \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[rr, bend left=40, "f"] \arrow[rr, bend left=25, "g"] \arrow[dr, "id"'] & W \arrow[r, "h"] \arrow[d, "s"] & Y \\
& X &
\end{tikzcd}$$

s a real equivalence.

full & faithful

$$\mathcal{D}[X, W] \xrightarrow{s_*} \mathcal{D}[X, X]$$

$$\exists! \; \hat{c} : d_1 \Rightarrow d_0 \;/\; s\hat{c} = id : \underbrace{id}_{sd_1} \Rightarrow \underbrace{id}_{sd_0}$$

define $\hat{H} = h\hat{c} : \underbrace{f}_{hd_1} \Rightarrow \underbrace{g}_{hd_0}$

given $\mathcal{C} \xrightarrow{F} \mathcal{D}$ 2 Functor and $H : f \rightsquigarrow g$

$H = (C, h)$. Then $FH = (FC, Fh) : Ff \rightsquigarrow Fg$

Now Fs is a real equivalence:

$$
\begin{tikzcd}
FX \arrow[r, shift left, "Fd_1"] \arrow[r, shift right, "Fd_0"'] \arrow[dr, "id"'] & FW \arrow[r, "Fh"] \arrow[d, "Fs"] & Y \\
 & FX &
\end{tikzcd}
$$

get $\widehat{FH} = Fh\,\widehat{FC} : Ff \Rightarrow Fg$

where $\widehat{FC} : Fd_1 \Rightarrow Fd_0$ UNIQUE $\mid Fs\,\widehat{FC} = id$

$$\boxed{\text{Define} \quad F[H] =_{def} \widehat{FH}}$$

where $[H]$ is equivalence class of "$\underbrace{\sim}_{\text{germ relation}}$"

Easy to prove $\boxed{H \sim H' \Rightarrow \widehat{FH} = \widehat{FH'}}$

so well defined.

But still not have a 2-category.

At this point Szyld's idea :

a) Consider a <u>generalization</u> of Quillen's cylinders.

b) <u>Forget</u> Model Categories and consider only a single class Σ (containing the identities) in a category $\mathcal{A}$

$$\begin{tikzcd}
X \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[rr, bend left=40, "f"] \arrow[rr, bend left=25, "g"'] \arrow[dr, "x"'] & W \arrow[r, "h"] \arrow[d, "s", "\circ"'] & Y \\
& Z &
\end{tikzcd}$$

Cylinder C

Homotopy $H=(C,h)$

$H: f \rightsquigarrow g$

As before

s equivalence $\Rightarrow$

$$\boxed{\begin{array}{l} \exists!\ \hat{c} : d_1 \Rightarrow d_0 \mid s\hat{c} = x \\ \hat{H} = h\hat{c} : \underset{\widehat{hd_1}}{f} \Rightarrow \underset{\widehat{hd_0}}{g} \end{array}}$$

given $\mathcal{A} \xrightarrow{F} \mathcal{D}$ define

$$\boxed{\begin{array}{l} F[H] : Ff \Rightarrow Fg \text{ in } \mathcal{D} \\ F[H] = \widehat{FH} \end{array}}$$

With this we decided <u>Forget</u> germ relation (for the moment)

Define Ad Hoc relation

$$\boxed{H \sim H' \iff \widehat{FH} = \widehat{FH'} \quad \forall F}$$

Things become simpler.

Horizontal Composition $\xrightarrow{l} \overset{H}{\rightsquigarrow} \xrightarrow{r}$

$$
\begin{tikzcd}
X' \arrow[r, "l"] \arrow[drr, bend right, "xl"'] & X \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[rr, bend left=50, "f"] \arrow[rr, bend left=30, "g"'] \arrow[dr, "x"] & W \arrow[r, "h"] \arrow[d, "s"] & Y \arrow[r, "r"] & Y' \\
 & & Z & &
\end{tikzcd}
$$

$H = (c, h) : f \rightsquigarrow g$

$Hl = (cl, h) : fl \rightsquigarrow gl$ (same h)

$rH = (c, rh) : rf \rightsquigarrow rg$ (same c)

Example of simple reasoning with the Ad Hoc relation

$\boxed{\widehat{Hl} = \hat{H} l}$

$\widehat{cl} = \hat{c} l$: $s(\hat{c} l) = (s\hat{c}) l = xl$

then $\widehat{Hl} = h \widehat{cl} = h \hat{c} l = \hat{H} l$

Define $\boxed{[H] l = [Hl]}$. Well defined:

$$\widehat{F(Hl)} = \widehat{FHFl} = \widehat{FH}\, Fl = \widehat{FH'}\, Fl = \widehat{FH'Fl} = \widehat{F(H'l)}.$$

Composition with r even easier because H and rH have same cylinder.

Vertical Composition

Forget vertical composition of homotopies.

Take 2-cells as sequences of composable homotopies

$f \overset{H}{\rightsquigarrow} g \overset{K}{\rightsquigarrow} t \qquad f \overset{H'}{\rightsquigarrow} g' \overset{K'}{\rightsquigarrow} t \qquad g \neq g'$

$$\boxed{(K,H) \sim (K'H') \iff \widehat{FK} \circ \widehat{FH} = \widehat{FK'} \circ \widehat{FH'} \text{ in } \mathcal{D}}$$

Define $[K,H]\,l = [Kl, Hl]$. well defined with no problem.

$r[K,H] = [rK, rH]$

Identities for horizontal composition same as $\mathcal{A}$.

Identities for vertical composition : $X \xrightarrow{f} Y$

$$\begin{tikzcd} X \arrow[r, shift left, "id"] \arrow[r, shift right, "id"'] \arrow[dr, "id"'] & X \arrow[r, "f"] \arrow[d, "id"] & Y \\ & X & \end{tikzcd} \sim \begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "f"'] \arrow[dr, "f"'] & Y \arrow[r, "id"] \arrow[d, "id"] & Y \\ & Y & \end{tikzcd}$$

Axioms of 2-category follow (apply F) from axioms of 2-category of $\mathcal{D}$ and using equations

$\widehat{Hl} = \hat{H}\, l$ and $\widehat{rH} = r\hat{H}$.

Have $\mathcal{C} \xrightarrow{\ i\ } \widetilde{Ho}(\mathcal{C})$ $\qquad iX = X$, $if = f$

Universal Property

$F(\Sigma) \subset$ Equivalences

$$\begin{tikzcd} \mathcal{A} \arrow[r, "i"] \arrow[dr, "F"'] \arrow[dr, phantom, shift left=3ex, "="] & \widetilde{Ho}(\mathcal{A}) \arrow[d, dashed, "\exists!\, \tilde F"] \\ & \mathcal{D} \end{tikzcd}$$

$\tilde F X = FX \qquad \tilde F f = Ff$

define
$$\boxed{\begin{aligned} \tilde F([H]) &= \widehat{FH} \\ \tilde F([H,K]) &= \widehat{FH} \circ \widehat{FK} \end{aligned}}$$

By definition is 2-functor for vertical composition

Using equations $\widehat{Hl} = \hat{H} l$ and $\widehat{rH} = r\hat{H}$ it follows functoriality for horizontal composition.

Uniqueness :

given

$$\begin{tikzcd} \mathcal{A} \arrow[r, "i"] \arrow[dr, "F"'] & \widetilde{\mathrm{H}}_0(\mathcal{A}) \arrow[d, "R"] \arrow[dl, phantom, "\cong"] \\ & \mathcal{D} \end{tikzcd}$$

$R i = F$

then $R([H]) = \widehat{FH}$

Can also prove the 2-categorical aspect :

given $F \overset{\eta}{\Longrightarrow} G$ define $\tilde{F} \overset{\tilde{\eta}}{\Longrightarrow} \tilde{G}$

$FX \xrightarrow{\eta X} GX$ $\quad$ $\tilde{F}X \xrightarrow{\tilde{\eta}X} \tilde{G}X$

$\boxed{\tilde{\eta}X = \eta X}$

Conditions of 2-naturality for $\tilde{\eta}$ hold (not easy)

ONLY REMAINS TO PROVE THAT i SENDS THE CLASS Σ TO EQUIVALENCES

FACT All 2-cells of $\widetilde{Ho}(\mathcal{A})$ are invertible

C, define C^{-1} by switching d_1 and d_0

$H=(C,h)$ define $H^{-1}=(C^{-1},h)$

$f \overset{H}{\rightsquigarrow} g$ clearly $g \overset{H^{-1}}{\rightsquigarrow} f$

Can easily check $[H]^{-1} = [H^{-1}]$

It follows:

Σ /
i) 3×2 property
(Joyal's terminology, pay 2 get 3)
ii) split generated.

then $i\Sigma \subset$ equivalences

Theorem

$$Hom(\widetilde{Ho}(\mathcal{A}), \mathcal{D}) \xrightarrow{i^*} Hom_+(\mathcal{A}, \mathcal{D})$$

isomorphism of categories.

RECOVERING QUILLEN

Can show that germ relation $\Rightarrow$ ad hoc relation

This is useful for some computations in what follows:

$\mathcal{A} = \mathcal{C}_{fc} \subset \mathcal{C}$ $\quad$ $\mathcal{C}$ a model category

$\Sigma = \mathcal{W}$. Conditions i), ii) hold.

Consider Quillen's cylinders.

Lemma X, Y fibrant-cofibrant $\quad X \underset{g}{\overset{f}{\rightrightarrows}} Y$

$H = (C, h) : f \rightsquigarrow g$

Then $\exists \; H' = (C', h') : f \rightsquigarrow g$

with C' a Quillen cylinder in $\mathcal{C}_{cf}$

and $H' \sim H$, $[H'] = [H]$

Lemma given $[H, K]$ $\quad f \overset{K}{\rightsquigarrow} g \overset{H}{\rightsquigarrow} t$

$\exists \; f \overset{N}{\rightsquigarrow} t \;/\; [N] = [H, K]$. IN PARTICULAR

TAKE π_0 and obtain Quillen's result.