Aspects of Descent via Bilimits

Fernando Lucatelli Nunes[1]

[1] Centre for Mathematics, University of Coimbra

Category Theory - CT 2018, University of Azores

Setting

$$\mathcal{A} : \mathfrak{C}^{\mathrm{op}} \to \mathbf{CAT},\ p \in Mor(\mathfrak{C})$$

[Janelidze and Tholen 1997]
Facets of descent II

Setting

$$\mathcal{A} : \mathfrak{C}^{\mathrm{op}} \to \mathbf{CAT},\ p \in Mor(\mathfrak{C})$$

Mains constructions:

- The descent category $\mathrm{Desc}_{\mathcal{A}}(p)$;
- The category of (Eilenberg Moore) algebras of the monad induced by $\mathcal{A}(p)! \dashv \mathcal{A}(p)$.

[Janelidze and Tholen 1997]
Facets of descent II

Setting

$$\mathcal{A} : \mathfrak{C}^{\mathrm{op}} \to \mathbf{CAT},\ p \in Mor(\mathfrak{C})$$

Mains constructions:

- The descent category $\mathrm{Desc}_{\mathcal{A}}(p)$;
- The category of (Eilenberg Moore) algebras of the monad induced by $\mathcal{A}(p)! \dashv \mathcal{A}(p)$.

[Ross Street 1976]
Limits indexed by category-valued 2-functors

[Ross Street 1980]
Correction to: “Fibrations in bicategories”

[Ross Street 2004]
Categorical and combinatorial aspects of descent theory

Aim

Work

The original idea was to investigate whether formal methods and commuting properties of (weighted) bilimits are useful in proving theorems of descent theory in the context of Facets of Descent II.

[Lucatelli Nunes 2018]
Pseudo-Kan Extensions and Descent Theory

Aim

Work

The original idea was to investigate whether formal methods and commuting properties of (weighted) bilimits are useful in proving theorems of descent theory in the context of Facets of Descent II.

Talk

Give an idea of the work, giving an overview of some results, including the approach to understand Descent vs Monadicity (Bénabou Roubaud Theorem).

[Lucatelli Nunes 2018]
Pseudo-Kan Extensions and Descent Theory

Outline

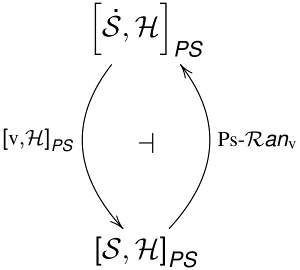
Right pseudo-Kan extension

$$\mathrm{v} : \mathcal{S} \to \dot{\mathcal{S}}, \quad \text{2-category } \mathcal{H}$$

$$\left[\dot{\mathcal{S}}, \mathcal{H}\right]_{PS} \xrightarrow{[\mathrm{v},\mathcal{H}]_{PS}} [\mathcal{S}, \mathcal{H}]_{PS}$$

Right pseudo-Kan extension

$$\mathrm{v} : \mathcal{S} \to \dot{\mathcal{S}}, \quad \text{2-category } \mathcal{H}$$



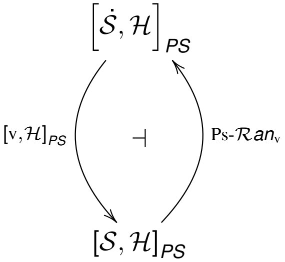
$$\varepsilon \;:\; [\mathrm{v}, \mathcal{H}]_{PS} \circ \mathrm{Ps}\text{-}\mathcal{R}an_{\mathrm{v}} \to \mathrm{Id}$$
$$\eta \;:\; \mathrm{Id} \to \mathrm{Ps}\text{-}\mathcal{R}an_{\mathrm{v}} \circ [\mathrm{v}, \mathcal{H}]_{PS}$$

$$s \;:\; \mathrm{Id}_L \cong (\varepsilon L) \circ (L\eta)$$
$$t \;:\; (U\varepsilon) \circ (\eta U) \cong \mathrm{Id}_U$$

plus coherence

Right pseudo-Kan extension

$$\mathrm{v} : \mathcal{S} \to \dot{\mathcal{S}}, \quad \text{2-category } \mathcal{H}$$



$$\varepsilon : [\mathrm{v}, \mathcal{H}]_{PS} \circ \mathrm{Ps}\text{-}\mathcal{R}an_{\mathrm{v}} \to \mathrm{Id}$$
$$\eta : \mathrm{Id} \to \mathrm{Ps}\text{-}\mathcal{R}an_{\mathrm{v}} \circ [\mathrm{v}, \mathcal{H}]_{PS}$$

$$s : \mathrm{Id}_L \cong (\varepsilon L) \circ (L\eta)$$
$$t : (U\varepsilon) \circ (\eta U) \cong \mathrm{Id}_U$$

plus coherence

v-effective

$\mathcal{D} : \dot{\mathcal{S}} \to \mathcal{H}$ is v-effective/exact if $\eta_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathrm{Ps}\text{-}\mathcal{R}an_{\mathrm{v}}(\mathcal{D} \circ \mathrm{v})$ is a pseudonatural equivalence.

Factorization and Comparison

$$\text{f.f. } \mathrm{v} : \mathcal{S} \to \dot{\mathcal{S}},\ \mathrm{Obj}(\dot{\mathcal{S}}) = \{\mathbf{e}\} \cup \mathrm{Obj}(\mathcal{S}), \mathcal{D} : \dot{\mathcal{S}} \to \mathcal{H}$$

- v-comparison: $\eta_{\mathcal{D}\mathbf{e}} : \mathcal{D}(\mathbf{e}) \to \mathrm{Ps}\text{-}\mathcal{R}an_{\mathrm{v}}\,(\mathcal{D} \circ \mathrm{v})\,(\mathbf{e})$
- v-“factorizations”:

For each morphism $f : \mathbf{e} \to \mathbf{a}$ of $\dot{\mathcal{S}}$,

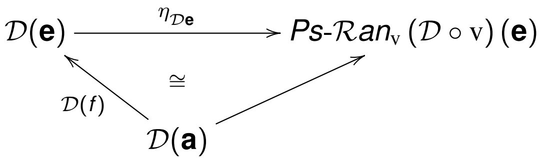
$$\begin{array}{ccc} \mathcal{D}(\mathbf{e}) & \xrightarrow{\ \eta_{\mathcal{D}\mathbf{e}}\ } & \mathrm{Ps}\text{-}\mathcal{R}an_{\mathrm{v}}\,(\mathcal{D} \circ \mathrm{v})\,(\mathbf{e}) \\ & \cong & \\ {\scriptstyle \mathcal{D}(f)} \searrow & & \swarrow \\ & \mathcal{D}(\mathbf{a}) & \end{array}$$

Factorization and Comparison

$$\text{f.f. v} : \mathcal{S} \to \dot{\mathcal{S}},\ \mathrm{Obj}(\dot{\mathcal{S}}) = \{\mathbf{e}\} \cup \mathrm{Obj}(\mathcal{S}), \mathcal{D} : \dot{\mathcal{S}} \to \mathcal{H}$$

- v-comparison: $\eta_{\mathcal{D}\mathbf{e}} : \mathcal{D}(\mathbf{e}) \to \text{Ps-}\mathcal{R}an_{\text{v}}(\mathcal{D} \circ \text{v})(\mathbf{e})$
- v-“factorizations”:

For each morphism $f : \mathbf{a} \to \mathbf{e}$ of $\dot{\mathcal{S}}$,



Pointwise Pseudo-Kan extension

Theorem

Given a pseudofunctor $\mathrm{D} : \mathcal{S} \to \mathcal{H}$,

$$\mathrm{Ps}\text{-}\mathcal{R}an_{\mathrm{v}}\mathrm{D}(\mathbf{a}) = \left\{ \dot{\mathcal{S}}(\mathbf{a}, \mathrm{v}-), \mathrm{D} \right\}_{\mathrm{bi}},$$

provided that these weighted bilimits exist in $\mathcal{H}$.

Pointwise Pseudo-Kan extension

Theorem

$$\text{Ps-}\mathcal{R}an_{\mathrm{v}}\mathrm{D}(\mathbf{a}) = \left\{\dot{\mathcal{S}}(\mathbf{a}, \mathrm{v}-), \mathrm{D}\right\}_{\mathrm{bi}}$$

Consequence

f.f. $\mathrm{v} : \mathcal{S} \to \dot{\mathcal{S}}$, $\mathrm{Obj}(\dot{\mathcal{S}}) = \{\mathbf{e}\} \cup \mathrm{Obj}(\mathcal{S})$, $\mathcal{D} : \dot{\mathcal{S}} \to \mathcal{H}$

$\mathcal{D}$ is v-effective
if and only if
$\mathcal{D}(\mathbf{e}) \to \left\{\dot{\mathcal{S}}(\mathbf{e}, \mathrm{v}-), \mathrm{D}\right\}_{\mathrm{bi}}$ is an equivalence.

Diagram of effective diagrams

$$\begin{aligned} \text{f.f. } \mathrm{v} : \mathcal{S} &\to \dot{\mathcal{S}} \\ \mathrm{Obj}(\dot{\mathcal{S}}) &= \{\mathbf{e}\} \cup \mathrm{Obj}(\mathcal{S}) \end{aligned} \qquad \begin{aligned} \text{f.f. } \mathrm{j} : \mathcal{R} &\to \dot{\mathcal{R}} \\ \mathrm{Obj}(\dot{\mathcal{R}}) &= \{\mathbf{z}\} \cup \mathrm{Obj}(\mathcal{R}) \end{aligned}$$

Theorem (Basic commuting property)

Given a pseudofunctor $M : \dot{\mathcal{S}} \to \left[\dot{\mathcal{R}}, \mathcal{H}\right]_{PS}$

- The image of $M \circ \mathrm{v} : \mathcal{S} \to \left[\dot{\mathcal{R}}, \mathcal{H}\right]_{PS}$ has only j-effective diagrams;
- Every diagram in the image of the mate $\widehat{M} : \dot{\mathcal{R}} \to \left[\dot{\mathcal{S}}, \mathcal{H}\right]_{PS}$ is v-effective

Then $M(\mathbf{e}) : \dot{\mathcal{R}} \to \mathcal{H}$ is j-effective as well.

Diagram of effective diagrams

$$\mathrm{Obj}(\dot{\mathcal{S}}) = \{\mathbf{e}\} \cup \mathrm{Obj}(\mathcal{S}) \qquad \mathrm{Obj}(\dot{\mathcal{R}}) = \{\mathbf{z}\} \cup \mathrm{Obj}(\mathcal{R})$$

Theorem (Basic commuting property)

Given a pseudofunctor $M : \dot{\mathcal{S}} \to \left[\dot{\mathcal{R}}, \mathcal{H}\right]_{PS}$

- The image of $M \circ \mathrm{v} : \mathcal{S} \to \left[\dot{\mathcal{R}}, \mathcal{H}\right]_{PS}$ has only j-effective diagrams;
- Every diagram in the image of the mate $\widehat{M} : \dot{\mathcal{R}} \to \left[\dot{\mathcal{S}}, \mathcal{H}\right]_{PS}$ is v-effective

Then $M(\mathbf{e}) : \dot{\mathcal{R}} \to \mathcal{H}$ is j-effective as well.

Comment

This very basic result and further non-basic results on commutativity of bilimits of the paper are consequences of 2-dimensional versions of Dubuc's adjoint triangle theorem proved therein.

Corollary

$$\mathrm{Obj}(\dot{\mathcal{S}}) = \{\mathbf{e}\} \cup \mathrm{Obj}(\mathcal{S}) \qquad \mathrm{Obj}(\dot{\mathcal{R}}) = \{\mathbf{z}\} \cup \mathrm{Obj}(\mathcal{R})$$

Corollary (Basic commuting property)

Given a pseudofunctor $M : \dot{\mathcal{S}} \to \left[\dot{\mathcal{R}}, \mathcal{H}\right]_{PS}$, we consider its mate $\widehat{M} : \dot{\mathcal{R}} \to \left[\dot{\mathcal{S}}, \mathcal{H}\right]_{PS}$.

- The image of $M \circ \mathrm{v} : \mathcal{S} \to \left[\dot{\mathcal{R}}, \mathcal{H}\right]_{PS}$ has only j-effective diagrams;
- The image of $\widehat{M} \circ \mathrm{j} : \mathcal{R} \to \left[\dot{\mathcal{S}}, \mathcal{H}\right]_{PS}$ has only v-effective diagrams;

Then $M(\mathbf{e}) : \dot{\mathcal{R}} \to \mathcal{H}$ is j-effective iff $\widehat{M}(\mathbf{z}) : \dot{\mathcal{S}} \to \mathcal{H}$ is v-effective.

Corollary

$$\mathrm{Obj}(\dot{\mathcal{S}}) = \{\mathbf{e}\} \cup \mathrm{Obj}(\mathcal{S}) \qquad \mathrm{Obj}(\dot{\mathcal{R}}) = \{\mathbf{z}\} \cup \mathrm{Obj}(\mathcal{R})$$

Corollary (Basic commuting property)

Given a pseudofunctor $M : \dot{\mathcal{S}} \to \left[\dot{\mathcal{R}}, \mathcal{H}\right]_{PS}$, we consider its mate $\widehat{M} : \dot{\mathcal{R}} \to \left[\dot{\mathcal{S}}, \mathcal{H}\right]_{PS}$.

- The image of $M \circ \mathrm{v} : \mathcal{S} \to \left[\dot{\mathcal{R}}, \mathcal{H}\right]_{PS}$ has only j-effective diagrams;
- The image of $\widehat{M} \circ \mathrm{j} : \mathcal{R} \to \left[\dot{\mathcal{S}}, \mathcal{H}\right]_{PS}$ has only v-effective diagrams;

Then $M(\mathbf{e}) : \dot{\mathcal{R}} \to \mathcal{H}$ is j-effective iff $\widehat{M}(\mathbf{z}) : \dot{\mathcal{S}} \to \mathcal{H}$ is v-effective.

Proof

- $\widehat{M}$ satisfies the hypotheses of the Theorem (on commuting properties of bilimits);
- Reciprocally, M satisfies the hypotheses of the Theorem.

The 2-category $\mathcal{A}dj$

Free Adjunction (Street and Schanuel)

We denote by $\mathcal{A}dj$ the 2-category generated by the diagram

$$\mathbf{e} \underset{f}{\overset{u}{\rightleftarrows}} \mathbf{a}$$

with 2-cells

$$\begin{aligned} \eta &: \mathrm{Id}_{\mathbf{a}} \to uf \\ \varepsilon &: fu \to \mathrm{Id}_{\mathbf{e}} \end{aligned}$$

satisfying the triangular identities.

We define the full inclusion $\mathrm{m} : \mathrm{Mnd} \to \mathcal{A}dj$, with $\mathrm{Obj}(\mathrm{Mnd}) = \{\mathbf{a}\}$

[S. Schanuel and R. Street 1986]
The Free Adjunction

Eilenberg Moore Factorization

Each adjunction in a 2-category $\mathcal{H}$ corresponds to a diagram

$$\mathcal{D} : \mathcal{A}dj \to \mathcal{H}.$$

The m-factorization gives the Eilenberg Moore factorization (if $\mathcal{H}$ is bicategorically complete) and the Eilenberg Moore comparison 1-cell.

$$\begin{array}{ccc} \mathcal{D}(\mathbf{e}) & \longrightarrow & Ps\text{-}\mathcal{R}an_{\mathrm{v}}\,(\mathcal{D}\circ \mathrm{v})\,(\mathbf{e}) \\ & \cong & \\ {\scriptstyle \mathcal{D}(u)} \searrow & & \swarrow \\ & \mathcal{D}(\mathbf{a}) & \end{array}$$

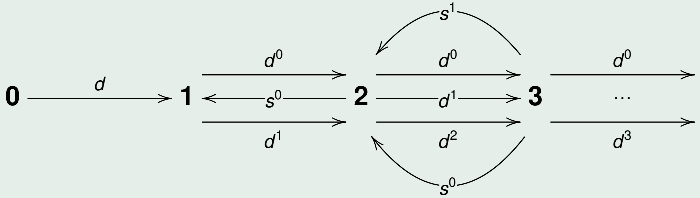
Thereby $\mathcal{D}$ is m-effective if and only if the right adjoint $\mathcal{D}(u)$ is monadic.

[S. Schanuel and R. Street 1986]
The Free Adjunction

The category Δ

Definition $\dot{\Delta}$

We denote by $\dot{\Delta}$ the category of finite ordinals and order-preserving functions

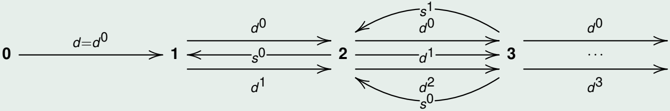


We define the full inclusion $\mathrm{g} : \Delta \to \dot{\Delta}$, with $\mathrm{Obj}(\dot{\Delta}) = \mathrm{Obj}(\Delta) \cup \{\mathbf{0}\}$

The category Δ

Definition

We denote by $\dot{\Delta}$ the category of finite ordinals and order-preserving functions



Full inclusion $\mathrm{g} : \Delta \to \dot{\Delta}$, with $\mathrm{Obj}(\dot{\Delta}) = \mathrm{Obj}(\Delta) \cup \{\mathbf{0}\}$

Coherence Theorem (Descent Object)

$$D : \Delta \to \mathcal{H}$$

$\mathrm{Ps}\text{-}\mathcal{R}an_{\mathrm{g}} D(\mathbf{0})$ is indeed the descent object of

$$\mathcal{D}(1) \rightleftarrows \mathcal{D}(2) \rightrightarrows \mathcal{D}(3)$$

First Lemma

Lemma on pseudonatural transformations

$$(\alpha : \mathcal{D} \longrightarrow \mathcal{D}') : \dot{\Delta} \to \mathcal{H}$$

- α is pointwise right adjoint in $\mathcal{H}$;
- α satisfies Beck-Chevalley condition;
- α_i is monadic in $\mathcal{H}$, $\forall i > 0$;
- $\mathcal{D}'$ is $\mathfrak{g}$-effective.

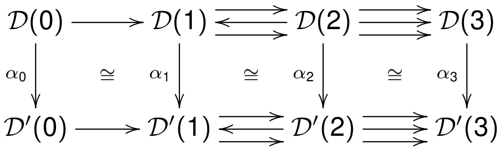
$\Longrightarrow$ α_0 is monadic if and only if $\mathcal{D}$ is $\mathfrak{g}$-effective.

First Lemma

Lemma on pseudonatural transformations

- α is pointwise right adjoint in $\mathcal{H}$;
- α satisfies Beck-Chevalley condition;
- α_i is monadic in $\mathcal{H}$, $\forall i > 0$;
- $\mathcal{D}'$ is $\mathfrak{g}$-effective.

$\Longrightarrow \alpha_0$ is monadic if and only if $\mathcal{D}$ is $\mathfrak{g}$-effective.



(Hypotheses of the) First Lemma

Hypotheses

- α is pointwise right adjoint in $\mathcal{H}$;
- α satisfies Beck-Chevalley condition;
- α_i is monadic in $\mathcal{H}$, $\forall i > 0$;
- $\mathcal{D}'$ is $\mathfrak{g}$-effective.

$$\begin{array}{ccccccc}
\mathcal{D}(0) & \longrightarrow & \mathcal{D}(1) & \rightleftarrows & \mathcal{D}(2) & \rightrightarrows & \mathcal{D}(3) \\
\downarrow \alpha_0 & \cong & \downarrow \alpha_1 & \cong & \downarrow \alpha_2 & \cong & \downarrow \alpha_3 \\
\mathcal{D}'(0) & \longrightarrow & \mathcal{D}'(1) & \rightleftarrows & \mathcal{D}'(2) & \rightrightarrows & \mathcal{D}'(3)
\end{array}$$

⇒ $M : \mathcal{A}dj \to \left[\dot{\Delta}, \mathcal{H}\right]_{PS}$

(Hypotheses of the) First Lemma

Hypotheses

- α is pointwise right adjoint in $\mathcal{H}$;
- α satisfies Beck-Chevalley condition;
- α_i is monadic in $\mathcal{H}$, $\forall i > 0$;
- $\mathcal{D}'$ is g-effective.

$$
\begin{array}{ccccccc}
\mathcal{D}(0) & \longrightarrow & \mathcal{D}(1) & \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} & \mathcal{D}(2) & \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} & \mathcal{D}(3) \\
\alpha_0 \downarrow & \cong & \alpha_1 \downarrow & \cong & \alpha_2 \downarrow & \cong & \alpha_3 \downarrow \\
\mathcal{D}'(0) & \longrightarrow & \mathcal{D}'(1) & \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} & \mathcal{D}'(2) & \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} & \mathcal{D}'(3)
\end{array}
$$

⇒ All the diagrams in the image of $\widehat{M} \circ \mathrm{g} : \Delta \to [\mathcal{A}dj, \mathcal{H}]_{PS}$ are m-effective.

(Hypotheses of the) First Lemma

Hypotheses

- α is pointwise right adjoint in $\mathcal{H}$;
- α satisfies Beck-Chevalley condition;
- α_i is monadic in $\mathcal{H}$, $\forall i > 0$;
- $\mathcal{D}'$ is g-effective.

$$
\begin{array}{ccccccc}
\mathcal{D}(0) & \longrightarrow & \mathcal{D}(1) & \rightleftarrows & \mathcal{D}(2) & \rightrightarrows & \mathcal{D}(3) \\
\downarrow \alpha_0 & \cong & \downarrow \alpha_1 & \cong & \downarrow \alpha_2 & \cong & \downarrow \alpha_3 \\
\mathcal{D}'(0) & \longrightarrow & \mathcal{D}'(1) & \rightleftarrows & \mathcal{D}'(2) & \rightrightarrows & \mathcal{D}'(3)
\end{array}
$$

- ⇒ The image of $\widehat{M} \circ \mathrm{g} : \Delta \to [\mathcal{A}dj, \mathcal{H}]_{PS}$ has only m-effective diagrams.
- ⇒ The diagram in the image of $M \circ \mathrm{m} : \mathrm{Mnd} \to \left[\dot{\Delta}, \mathcal{H}\right]_{PS}$ is g-effective.

(Hypotheses of the) First Lemma

Hypotheses

- α is pointwise right adjoint in $\mathcal{H}$;
- α satisfies Beck-Chevalley condition;
- α_i is monadic in $\mathcal{H}$, $\forall i > 0$;
- $\mathcal{D}'$ is g-effective.

- The image of $\widehat{M} \circ \mathrm{g} : \Delta \to [\mathcal{A}dj, \mathcal{H}]_{PS}$ has only m-effective diagrams.
- The diagram in the image of $M \circ \mathrm{m} : \mathrm{Mnd} \to \left[\dot{\Delta}, \mathcal{H}\right]_{PS}$ is g-effective.

$\Longrightarrow$ $M(0)$ is m-effective (*i.e* α_0 is monadic) if and only if $\widehat{M}(\mathrm{e})$ is g-effective (*i.e* $\mathcal{D}$ is g-effective).

Consequence of the First Lemma

Recall that $\mathrm{su} : \dot{\Delta} \to \dot{\Delta}$ given by $(1 + -)$.

Consequence of the First Lemma

Recall that $\mathrm{su} : \dot{\Delta} \to \dot{\Delta}$ given by $(1 + -)$.

Lemma on pseudocosimplicial objects

- $\mathcal{D} : \dot{\Delta} \to \mathcal{H}$
- The invertible 2-cells of the pseudofunctor $\mathcal{D}$

$$\begin{array}{ccc} \mathcal{D}(n-1) & \xrightarrow{\mathcal{D}(d^{i-1})} & \mathcal{D}(n) \\ {\scriptstyle \mathcal{D}(d^0)}\downarrow & \cong & \downarrow{\scriptstyle \mathcal{A}(d^0)} \\ \mathcal{D}(n) & \xrightarrow[\mathcal{A}(d^i)]{} & \mathcal{D}(n+1) \end{array}$$

satisfy the Beck-Chevalley condition.

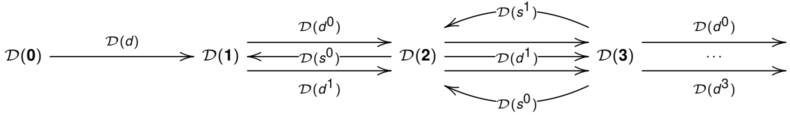
- $\mathcal{D} \circ \mathrm{su}$ is g-effective;
- $\mathcal{D}(d)$ and every $\mathcal{D}(d^0)$ have left adjoints;

$\Longrightarrow$ $\mathcal{D}(d)$ is monadic if and only if $\mathcal{D}$ is g-effective.

Hypotheses of the Lemma on pseudocosimplicial objects

Hypotheses of the Lemma on pseudocosimplicial objects

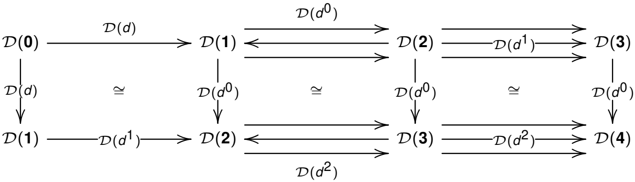
- Beck-Chevalley Condition plus the fact that $\mathcal{D}(d)$ and every $\mathcal{D}(d^0)$ have left adjoints;
- $\mathcal{D} \circ \mathrm{su}$ is g-effective;



Hypotheses of the Lemma on pseudocosimplicial objects

Hypotheses of the Lemma on pseudocosimplicial objects

- Beck-Chevalley Condition plus the fact that $\mathcal{D}(d)$ and every $\mathcal{D}(d^0)$ have left adjoints;
- $\mathcal{D} \circ \mathrm{su}$ is g-effective;

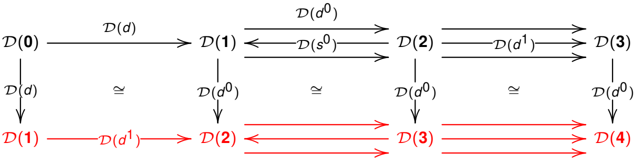


⇒ $\alpha : \mathcal{D} \longrightarrow \mathcal{D} \circ \mathrm{su}$

Hypotheses of the Lemma on pseudocosimplicial objects

Hypotheses of the Lemma on pseudocosimplicial objects

- Beck-Chevalley Condition plus the fact that $\mathcal{D}(d)$ and every $\mathcal{D}(d^0)$ have left adjoints;
- $\mathcal{D} \circ \mathrm{su}$ is g-effective;

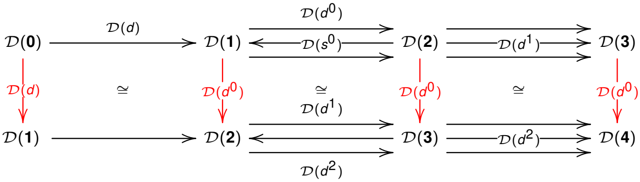


⇒ $\alpha : \mathcal{D} \longrightarrow \mathcal{D} \circ \mathrm{su}$

Hypotheses of the Lemma on pseudocosimplicial objects

Hypotheses of the Lemma on pseudocosimplicial objects

- Beck-Chevalley Condition plus the fact that $\mathcal{D}(d)$ and every $\mathcal{D}(d^0)$ have left adjoints;
- $\mathcal{D} \circ \mathrm{su}$ is g-effective;

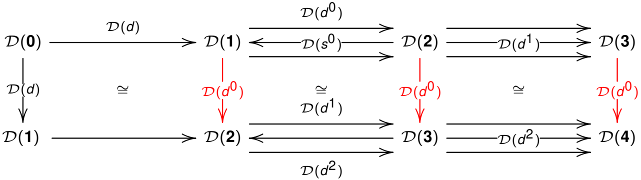


$\Rightarrow$ $\alpha : \mathcal{D} \longrightarrow \mathcal{D} \circ \mathrm{su}$ has a left adjoint in $\left[\dot{\Delta}, \mathcal{H}\right]_{PS}$.

Hypotheses of the Lemma on pseudocosimplicial objects

Hypotheses of the Lemma on pseudocosimplicial objects

- Beck-Chevalley Condition plus the fact that $\mathcal{D}(d)$ and every $\mathcal{D}(d^0)$ have left adjoints;
- $\mathcal{D} \circ \mathrm{su}$ is $\mathfrak{g}$-effective;

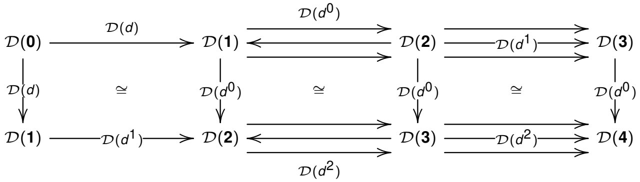


- $\alpha : \mathcal{D} \longrightarrow \mathcal{D} \circ \mathrm{su}$ has a left adjoint in $\left[\dot{\Delta}, \mathcal{H}\right]_{PS}$;
- α_i is monadic for all $i > 0$.

Hypotheses of the Lemma on pseudocosimplicial objects

Hypotheses of the Lemma on pseudocosimplicial objects

- Beck-Chevalley Condition plus the fact that $\mathcal{D}(d)$ and every $\mathcal{D}(d^0)$ have left adjoints;
- $\mathcal{D} \circ \mathrm{su}$ is g-effective;



- $\alpha : \mathcal{D} \longrightarrow \mathcal{D} \circ \mathrm{su}$ has a left adjoint in $\left[\dot{\Delta}, \mathcal{H}\right]_{PS}$;
- α_i is monadic for all $i > 0$.

$\Longrightarrow$ α_0 is monadic (*i.e.* $\mathcal{D}(d)$ is monadic) if and only if $\mathcal{D}$ is g-effective.

Context

1. $\mathfrak{C}$ with pullbacks;
2. $\mathcal{A} : \mathfrak{C}^{\mathrm{op}} \to \mathbf{CAT}$;
3. $\mathcal{A}(q)! \dashv \mathcal{A}(q)$;
4. $p \in \mathfrak{C}(E, B)$

- $$\dot{\Delta} \to \mathfrak{C}$$

$$E \times_B E \times_B E \times_B E \; \overset{\cdots}{\rightrightarrows} \; E \times_B E \times_B E \; \rightrightarrows \; E \times_B E \; \leftrightarrows \; E \xrightarrow{p} B$$

[Janelidze and Tholen]
Facets of descent II

Context

1. $\mathfrak{C}$ with pullbacks;
2. $\mathcal{A} : \mathfrak{C}^{\mathrm{op}} \to \mathbf{CAT}$;
3. $\mathcal{A}(q)! \dashv \mathcal{A}(q)$;
4. $p \in \mathfrak{C}(E, B)$

- $\mathcal{A}_p^{\mathcal{D}esc} : \dot{\Delta} \to \mathbf{CAT}$ (descent diagram induced by p)

$$\mathcal{A}(B) \xrightarrow{\mathcal{A}(p)} \mathcal{A}(E) \rightleftarrows \mathcal{A}(E \times_p E) \rightrightarrows \mathcal{A}(E \times_p E \times_p E)$$

[Janelidze and Tholen]
Facets of descent II

Context

1. $\mathfrak{C}$ with pullbacks;
2. $\mathcal{A} : \mathfrak{C}^{\mathrm{op}} \to \mathbf{CAT}$;
3. $\mathcal{A}(q)! \dashv \mathcal{A}(q)$;
4. $p \in \mathfrak{C}(E, B)$

Definition: $\mathcal{A}$-effective descent

p is of $\mathcal{A}$-effective descent if $\mathcal{A}_p^{\mathcal{D}esc} : \dot{\Delta} \to \mathbf{CAT}$ is $\mathfrak{g}$-effective.

- $\mathcal{A}_p^{\mathcal{D}esc} : \dot{\Delta} \to \mathbf{CAT}$ (descent diagram induced by p)

$$\mathcal{A}(B) \xrightarrow{\mathcal{A}(p)} \mathcal{A}(E) \rightleftarrows \mathcal{A}(E \times_p E) \rightrightarrows \mathcal{A}(E \times_p E \times_p E)$$

[Janelidze and Tholen]
Facets of descent II

Proof

Definition: $\mathcal{A}$-effective descent

p is of $\mathcal{A}$-effective descent if $\mathcal{A}_p^{\mathcal{D}esc} : \dot{\Delta} \to \mathbf{CAT}$ is $\mathfrak{g}$-effective.

Apply Lemma on pseudocosimplicial objects to:

$$\mathcal{A}(B) \xrightarrow{\mathcal{A}(p)} \mathcal{A}(E) \rightleftarrows \mathcal{A}(E \times_p E) \rightrightarrows \mathcal{A}(E \times_p E \times_p E)$$

Proof

Definition: $\mathcal{A}$-effective descent

p is of $\mathcal{A}$-effective descent if $\mathcal{A}_p^{\mathcal{D}esc} : \dot{\Delta} \to \mathbf{CAT}$ is $\mathfrak{g}$-effective.

Apply Lemma on pseudocosimplicial objects to:

$$\mathcal{A}(B) \xrightarrow{\mathcal{A}(p)} \mathcal{A}(E) \rightleftarrows \mathcal{A}(E \times_p E) \rightrightarrows \mathcal{A}(E \times_p E \times_p E)$$

- Beck-Chevalley is equivalent to the usual Beck-Chevalley condition;

Proof

Definition: $\mathcal{A}$-effective descent

p is of $\mathcal{A}$-effective descent if $\mathcal{A}_p^{\mathcal{D}esc} : \dot{\Delta} \to \mathbf{CAT}$ is $\mathfrak{g}$-effective.

Apply Lemma on pseudocosimplicial objects to:

$$\mathcal{A}(B) \xrightarrow{\mathcal{A}(p)} \mathcal{A}(E) \rightrightarrows \mathcal{A}(E \times_p E) \rightrightarrows \mathcal{A}(E \times_p E \times_p E)$$

- Beck-Chevalley is equivalent to the usual Beck-Chevalley condition;
- $\mathcal{A}_p^{\mathcal{D}esc} \circ \mathrm{su} \cong \mathcal{A}_{E\times_p E \to E}^{\mathcal{D}esc}$

Proof

Definition: $\mathcal{A}$-effective descent

p is of $\mathcal{A}$-effective descent if $\mathcal{A}_p^{\mathcal{D}esc} : \dot{\Delta} \to \mathbf{CAT}$ is $\mathfrak{g}$-effective.

Apply Lemma on pseudocosimplicial objects to:

$$\mathcal{A}(B) \xrightarrow{\mathcal{A}(p)} \mathcal{A}(E) \rightrightarrows \mathcal{A}(E \times_p E) \rightrightarrows \mathcal{A}(E \times_p E \times_p E)$$

- Beck-Chevalley is equivalent to the usual Beck-Chevalley condition;
- $\mathcal{A}_p^{\mathcal{D}esc} \circ \mathrm{su} \cong \mathcal{A}_{E\times_p E \to E}^{\mathcal{D}esc}$

Assumed Result

- Projections $E \times_p E \to E$ are of $\mathcal{A}$-effective descent (it is a direct a consequence of the fact that split epimorphisms are absolute $\mathcal{A}$-effective descent Facets of Descent II) ;

Proof

Definition: $\mathcal{A}$-effective descent

p is of $\mathcal{A}$-effective descent if $\mathcal{A}_p^{\mathcal{D}esc} : \dot{\Delta} \to \mathbf{CAT}$ is $\mathfrak{g}$-effective.

Apply Lemma on pseudocosimplicial objects to:

$$\mathcal{A}(B) \xrightarrow{\mathcal{A}(p)} \mathcal{A}(E) \rightrightarrows \mathcal{A}(E \times_p E) \rightrightarrows \mathcal{A}(E \times_p E \times_p E)$$

- Beck-Chevalley is equivalent to the usual Beck-Chevalley condition;

Assumed Result

- Projections $E \times_p E \to E$ are of $\mathcal{A}$-effective descent (it is a direct a consequence of the fact that split epimorphisms are absolute $\mathcal{A}$-effective descent Facets of Descent II) ;

- $\mathcal{A}_p^{\mathcal{D}esc} \circ \mathrm{su} \cong \mathcal{A}_{E\times_p E \to E}^{\mathcal{D}esc}$ is always $\mathfrak{g}$-effective

Proof

Definition: $\mathcal{A}$-effective descent

p is of $\mathcal{A}$-effective descent if $\mathcal{A}_p^{\mathcal{D}esc} : \dot{\Delta} \to$ **CAT** is $\mathfrak{g}$-effective.

Apply Lemma on pseudocosimplicial objects to:

$$\mathcal{A}(B) \xrightarrow{\mathcal{A}(p)} \mathcal{A}(E) \rightleftarrows\!\!\!\rightarrow \mathcal{A}(E \times_p E) \rightrightarrows\!\!\!\rightarrow \mathcal{A}(E \times_p E \times_p E)$$

- Beck-Chevalley is equivalent to the usual Beck-Chevalley condition;
- $\mathcal{A}_p^{\mathcal{D}esc} \circ \mathrm{su} \cong \mathcal{A}_{E \times_p E \to E}^{\mathcal{D}esc}$ is always $\mathfrak{g}$-effective

$\Rightarrow$ p is of $\mathcal{A}$-effective descent if and only if $\mathcal{A}(p)$ is monadic.

Consequences of results on commutativity of bilimits

Overview of Examples of Results

- Pseudopullback theorem

[Lucatelli Nunes]
Pseudo-Kan Extensions and Descent Theory

Consequences of results on commutativity of bilimits

Overview of Examples of Results

- Pseudopullback theorem

(Pseudopullback) Theorem

$\mathfrak{Q}, \mathfrak{C}, \mathbb{D}$ and $\mathfrak{E}$ be categories with pullbacks.

$$\begin{array}{ccc} \mathfrak{Q} & \xrightarrow{S} & \mathfrak{C} \\ {\scriptstyle Z}\downarrow & \cong & \downarrow{\scriptstyle F} \\ \mathbb{D} & \longrightarrow & \mathfrak{E} \end{array}$$

such that S, G, F and Z are pullback preserving functors. If p is a morphism in $\mathfrak{Q}$ such that $S(p), Z(p)$ are of effective descent w.r.t. the basic fibration and $FS(p)$ is of descent w.r.t. the basic fibration, then p is of effective descent.

[Lucatelli Nunes]
Pseudo-Kan Extensions and Descent Theory

Consequences of results on commutativity of bilimits

Overview of Examples of Results

- Pseudopullback theorem
  - Effective descent morphisms for categories of enriched categories satisfying suitable hypotheses;

[Lucatelli Nunes]
Pseudo-Kan Extensions and Descent Theory

Consequences of results on commutativity of bilimits

Overview of Examples of Results

- Pseudopullback theorem
  - Effective descent morphisms for categories of enriched categories satisfying suitable hypotheses;
- Results (including the classical ones) on reflection of effective morphisms by embeddings;

[Lucatelli Nunes]
Pseudo-Kan Extensions and Descent Theory

Consequences of results on commutativity of bilimits

Overview of Examples of Results

- Pseudopullback theorem
  - Effective descent morphisms for categories of enriched categories satisfying suitable hypotheses;
- Results (including the classical ones) on reflection of effective morphisms by embeddings;
- A “Galois” theorem of Janelidze-Schumacher-Street.

[Lucatelli Nunes]
Pseudo-Kan Extensions and Descent Theory

Enriched Categories

Theorem on Enriched Categories

Let V be a infinitary lextensive category such that there is a full inclusion $\mathbf{Set} \to V : X \mapsto X \otimes 1$.

$$\begin{array}{ccc} V - \mathbf{Cat} & \longrightarrow & \mathbf{Cat}(V) \\ \downarrow & \cong & \downarrow \\ \mathbf{Set} & \longrightarrow & V \end{array}$$

is a pseudopullback such that the arrows are pullback preserving functors.

Enriched Categories

Theorem on Enriched Categories

Let V be a infinitary lextensive category such that there is a full inclusion $\mathbf{Set} \to V : X \mapsto X \otimes 1$.

$$\begin{array}{ccc} V - \mathbf{Cat} & \longrightarrow & \mathbf{Cat}(V) \\ \downarrow & \cong & \downarrow \\ \mathbf{Set} & \longrightarrow & V \end{array}$$

is a pseudopullback such that the arrows are pullback preserving functors.

Corollary

Let V be as above and such that it has a regular epi-mono factorization. Then V-$\mathbf{Cat} \to \mathbf{Cat}(V)$ reflects effective descent morphisms.

Enriched Categories

Theorem on Enriched Categories

Let V be a infinitary lextensive category such that there is a full inclusion $\mathbf{Set} \to V : X \mapsto X \otimes 1$.

$$\begin{array}{ccc} V-\mathbf{Cat} & \longrightarrow & \mathbf{Cat}(V) \\ \downarrow & \simeq & \downarrow \\ \mathbf{Set} & \longrightarrow & V \end{array}$$

is a pseudopullback such that the arrows are pullback preserving functors.

Corollary

Let V be as above and such that it has a regular epi-mono factorization. Then V-$\mathbf{Cat} \to \mathbf{Cat}(V)$ reflects effective descent morphisms.

[Le Creurer 1999]

Descent of internal categories

Bénabou Roubaud Counterpart

Theorem

Let $\mathcal{H}$ be a 2-category of lax descent objects and comma colimits (*i.e.* the dual notion of comma objects).
A morphism $f : A \to B$ is monadic if and only if it gives the lax descent object of the *lax* cosimplicial object (higher cokernel)

$$B \underset{D_1}{\overset{D_0}{\rightleftarrows\!\!\!\rightarrow}} \lfloor p, p \rfloor \mathrel{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}} D_0 \amalg_B D_1$$

Bénabou Roubaud Counterpart

Theorem

f is monadic iff f gives the lax descent object of its higher cokernel. That is to say,

$$A \xrightarrow{f} B \underset{D_1}{\overset{D_0}{\rightleftarrows}} \lfloor p, p \rfloor \rightrightarrows D_0 \amalg_B D_1$$

is effective/exact (after defining a different domain category Δ).

Bénabou Roubaud Counterpart

Theorem

f is monadic iff f gives the lax descent object of its higher cokernel. That is to say,

$$A \xrightarrow{f} B \underset{D_1}{\overset{D_0}{\rightrightarrows}} \lfloor p, p \rfloor \;\Rrightarrow\; D_0 \amalg_B D_1$$

is effective/exact (after defining a different domain category Δ).

Comments

- I have a explicit proof of the result;

Bénabou Roubaud Counterpart

Theorem

f is monadic iff f gives the lax descent object of its higher cokernel. That is to say,

$$A \xrightarrow{f} B \underset{D_1}{\overset{D_0}{\rightrightarrows}} \lfloor p, p \rfloor \Rrightarrow D_0 \amalg_B D_1$$

is effective/exact (after defining a different domain category Δ).

Comments

- I have a explicit proof of the result;
- But I am working on a lax version of the lemma for pseudocosimplicial objects that actually gives the result above as a direct consequence;

Bénabou Roubaud Counterpart

Theorem

f is monadic iff f gives the lax descent object of its higher cokernel.

Comments

- I have a explicit proof of the result;
- But I am working on a lax version of the lemma for pseudocosimplicial objects that actually gives the result above as a direct consequence;
- I am mostly employing the techniques already introduced in the paper;

[Lucatelli Nunes, TAC, 2018]
Pseudo-Kan Extensions and Descent Theory

Bénabou Roubaud Counterpart

Theorem

f is monadic iff f gives the lax descent object of its higher cokernel.

Comments

- I have a explicit proof of the result;
- But I am working on a lax version of the lemma for pseudocosimplicial objects that actually gives the result above as a direct consequence;
- I am mostly employing the techniques already introduced in the paper;
- Finally, this lax version also implies in Bénabou Roubaud, putting Bénabou Roubaud and the result on Monadicity above as consequences of the very same version of the result on commutativity.

Thank you!