CT2018 - University of Azores

A categorical explanation of why Church's Thesis holds in the Effective Topos

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Arithmetic doctrines

- $P\colon \mathcal{C}^{op} \to \mathcal{H}eyt$
- $\mathcal{C}$ has finite products
- for $f\colon X \to Y$ the map $P(f)\colon P(Y) \to P(X)$ has (natural) a left and a right adjoint

$$\exists_f\colon P(X) \to P(Y) \qquad \forall_f\colon P(X) \to P(Y)$$

- $\mathcal{C}$ is weakly cartesian closed (wcc)
- $\mathcal{C}$ has a parametrized nno (pnno) $1 \xrightarrow{\mathbf{o}} \mathbf{N} \xrightarrow{\mathbf{s}} \mathbf{N}$
- P satisfies the induction principle on $\mathbf{N}$

Examples

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Subobjects

$\mathcal{C}$ is

- elementary topos
- nno

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Weak subobjects

$\mathcal{C}$ is

- lex
- finite co-products
- weakly lcc
- pnno

$$\Psi_{\mathcal{C}} \colon \mathcal{C}^{op} \to \mathcal{H}eyt$$

$$A \mapsto (\mathcal{C}/A)_{\mathsf{po}}$$

Internal language

A in $\mathcal{C}$	$f\colon X \to A$	$\alpha \in P(A)$	$P(f)(\alpha) \in P(X)$
$a\colon A$	$x\colon X \mid f(x)\colon A$	$a\colon A \mid \alpha(a)$	$x\colon X \mid \alpha(f(x))$

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$$\phi_1 \wedge \ldots \wedge \phi_n \leq \psi \quad \text{in} \quad P(A_1 \times \ldots \times A_k)$$

becomes

$$a_1\colon A_1, \ldots, a_k\colon A_k \mid \phi_1(a_1, \ldots, a_k), \ldots, \phi_n(a_1, \ldots, a_k) \vdash \psi(a_1, \ldots, a_k)$$

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$$\alpha = \top_A \text{ becomes } a \colon A \vdash_P \alpha(a)$$

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P has comprehensive diagonals if for all $f, g \colon A \to X$

$$f = g \quad \text{iff} \quad a \colon A \vdash_P f(a) =_X g(a)$$

Formal Church's Thesis

P is arithmetic. $\mathbf{N}^{\mathbf{N}}$ is a weak exp.

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Formal Church's Thesis (CT)

$$\vdash_P \forall_{x:\mathbf{N}} \exists_{y:\mathbf{N}} R(x,y) \to \exists_{e:\mathbf{N}} \forall_{x:\mathbf{N}} \exists_{y:\mathbf{N}} \left[\mathsf{T}(e,x,y) =_{\mathbf{N}} \mathbf{1} \wedge R(x, \mathsf{U}(y)) \right]$$

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Rule of choice (RC)

if $a\colon A \vdash_P \exists_{b:B} R(a,b)$, there is $f\colon A \to B$ s.t. $a\colon A \vdash_P R(a, f(a))$

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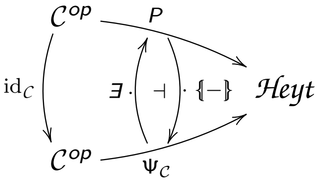
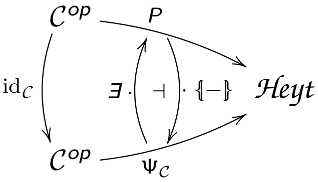
(TCT) + (RC) + full weak comprehension $\Rightarrow$ (CT)

Weak comprehension



Weak comprehension



Weak comprehension is full iff $\exists\{\!|-|\!\} = \mathrm{id}_P$.

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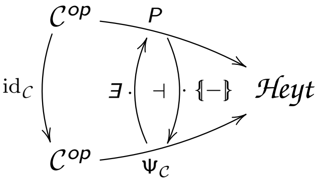

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Theorem: $\{\!|-|\!\}\exists = \mathrm{id}_{\Psi_{\mathcal{C}}}$ iff P satisfies (RC)

[Maietti, Pasquali, Rosolini. Tbilisi Mathematical Journal. 2017]

Elementary quotient completion

[Maietti, Rosolini. Elementary quotient completion. 2013]
[Maietti, Rosolini. Unifying exact completions. 2015]

Elementary quotient completion

$$\mathcal{C}^{op} \xrightarrow{P} \mathcal{H}eyt$$

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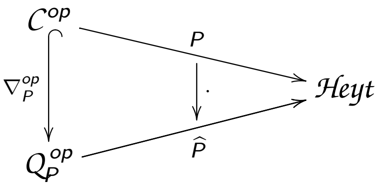
Elementary quotient completion



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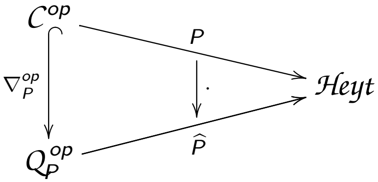

$\widehat{P}$ has effective quotients. $\widehat{P}$ is the free such on P.

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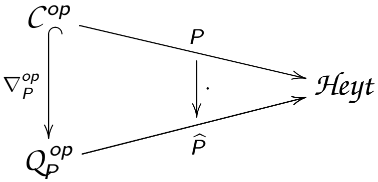

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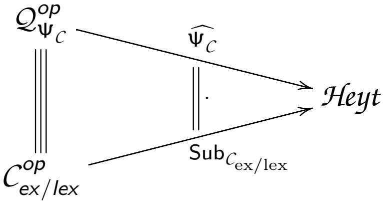

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$$\begin{array}{c} \widehat{P} \\ {\scriptstyle I_{\text{eqc}}}\big\uparrow \\ P \end{array} \qquad\qquad \begin{array}{c} \mathrm{Sub}_{\mathcal{C}_{\text{ex/lex}}} \\ \big\uparrow{\scriptstyle I_{\text{eqc}}} \\ \Psi_{\mathcal{C}} \end{array}$$

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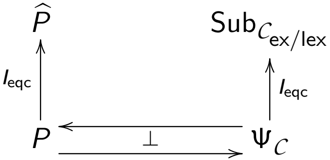

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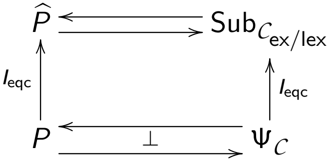

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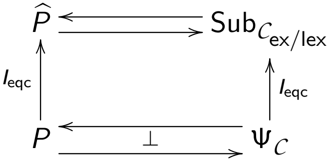

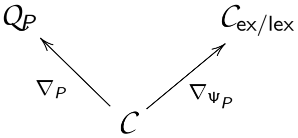

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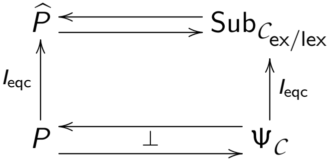

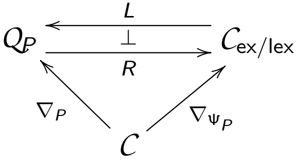

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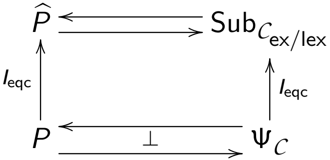

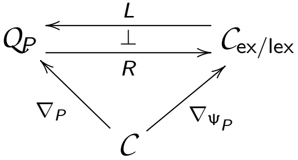

R is full and faithful
L preserves finite products

Elementary quotient completion and (TCT), (CT)

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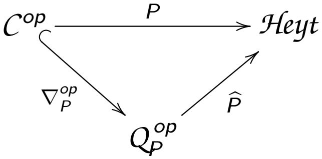
$\nabla_P(\mathbf{N})$ is a pnno in $\mathcal{Q}_P$



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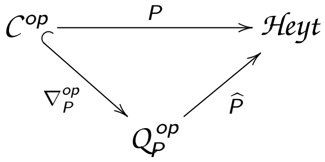

Theorem:

- P satisfies (TCT) if and only if $\widehat{P}$ satisfies (TCT)
- P satisfies (CT) if and only if $\widehat{P}$ satisfies (CT)

Full comprehension and (TCT), (CT)

$$P \underset{\{\!|-|\!\}}{\overset{\exists}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \Psi_{\mathcal{C}}$$

(RC) iff $\{\!|-|\!\}\exists = \mathrm{id}_{\Psi_{\mathcal{C}}}$

Full comprehension and (TCT), (CT)

$$P \underset{\{\!|-|\!\}}{\overset{\exists}{\underset{\perp}{\rightleftarrows}}} \Psi_{\mathcal{C}} \qquad \text{(RC) iff} \quad \{\!|-|\!\}\exists = \mathrm{id}_{\Psi_{\mathcal{C}}}$$

$$\alpha \leq \beta \text{ in } P(A) \quad \text{iff} \quad \{\!|\alpha|\!\} \leq \{\!|\beta|\!\} \text{ in } \Psi_{\mathcal{C}}(A)$$

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$$\begin{aligned}
\{\!|=_A|\!\} &= =_A \\
\{\!|P(f)(\alpha)|\!\} &= \Psi_{\mathcal{C}}(f)\{\!|\alpha|\!\} \\
\{\!|\alpha \wedge \beta|\!\} &= \{\!|\alpha|\!\} \wedge \{\!|\beta|\!\} \\
\{\!|\alpha \to \beta|\!\} &= \{\!|\alpha|\!\} \to \{\!|\beta|\!\} \\
\{\!|\forall_f \phi|\!\} &= \Pi_f \{\!|\phi|\!\}
\end{aligned}$$

Full comprehension and (TCT), (CT)

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\{\!|\forall_f \phi|\!\} &= \Pi_f \{\!|\phi|\!\} \\
\{\!|\alpha \vee \beta|\!\} &= \{\!|-|\!\}\exists\, [\{\!|\alpha|\!\} \vee \{\!|\beta|\!\}] \\
\{\!|\exists_f \phi|\!\} &= \{\!|-|\!\}\exists\, [\Sigma_f \{\!|\phi|\!\}]
\end{aligned}$$

Full comprehension and (TCT), (CT)

$R \in P(A \times B)$

R has a Skolem arrow for B if there is $f\colon A \to B$ s.t.

$$x\colon A \mid \exists_{y:B} R(x,y) \vdash R(x, f(x))$$

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Theorem: if R has a Skolem arrow for B

$$\{\!|\exists_\pi \phi|\!\} = \{\!|-|\!\}\exists\,[\Sigma_\pi \{\!|\phi|\!\}] = \Sigma_\pi \{\!|\phi|\!\}$$

where $\pi \colon A \times B \to A$, i.e.

$$\{\!|\exists_{y:B} \phi(x,y)|\!\} = \Sigma_{y:B} \{\!|\phi|\!\}(x,y)$$

Full comprehension and (TCT), (CT)

$$\text{(TCT)} \quad \forall_{f:\mathbf{N}^{\mathbf{N}}} \exists_{e:\mathbf{N}} \forall_{x:\mathbf{N}} \exists_{y:\mathbf{N}} \; [\mathrm{T}(e,x,y) =_{\mathbf{N}} \mathbf{1} \wedge \mathrm{U}(y) =_{\mathbf{N}} ev(x,f)]$$

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Suppose P has Skolem arrows:

$$\mathbf{N}^{\mathbf{N}} \times \mathbf{N} \times \mathbf{N} \to \mathbf{N} \qquad\qquad \mathbf{N}^{\mathbf{N}} \to \mathbf{N}$$

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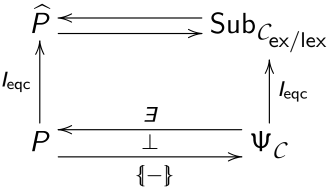

Full comprehension and (TCT), (CT)

$$\text{(TCT)}\qquad \forall_{f:\mathbf{N}^{\mathbf{N}}}\exists_{e:\mathbf{N}}\underbrace{\forall_{x:\mathbf{N}}\exists_{y:\mathbf{N}}\ \underbrace{[\mathrm{T}(e,x,y) =_{\mathbf{N}} \mathbf{1} \wedge \mathrm{U}(y) =_{\mathbf{N}} ev(x,f)]}_{}}_{}$$

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$$\mathbf{N}^{\mathbf{N}} \times \mathbf{N} \times \mathbf{N} \to \mathbf{N} \qquad\qquad \mathbf{N}^{\mathbf{N}} \to \mathbf{N}$$

$$\text{(TCT)}\qquad \begin{array}{ccc} \widehat{P} & \rightleftarrows & \mathrm{Sub}_{\mathcal{C}_{\mathrm{ex/lex}}} \\ \uparrow I_{\mathrm{eqc}} & & \uparrow I_{\mathrm{eqc}} \\ P & \underset{\{-\}}{\overset{\exists}{\underset{\rightarrow}{\overset{\leftarrow}{\bot}}}} & \Psi_{\mathcal{C}} \end{array}$$

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\end{array}$$

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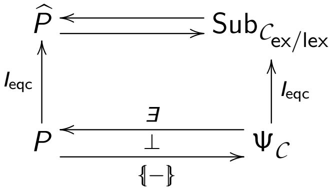
(TCT)



(TCT) (CT)

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(TCT) (CT)

Application to assemblies

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The category of $\mathcal{A}sm$

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objects: (A, α) where $\alpha \colon A \to \mathcal{P}ow_*(\mathbb{N})$

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objects: (A, α) where $\alpha \colon A \to \mathscr{Pow}_*(\mathbb{N})$

arrows: $f \colon (A, \alpha) \to (B, \beta)$ where $f \colon A \to B$ has a track

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objects: (A, α) where $\alpha\colon A \to \mathcal{P}ow_*(\mathbb{N})$

arrows: $f\colon (A, \alpha) \to (B, \beta)$ where $f\colon A \to B$ has a track, i.e.

there exists $n \in \mathbb{N}$, such that

for all $a \in A$ and all $p \in \alpha(a)$

$$\varphi_n(p)\downarrow \text{ and } \varphi_n(p) \in \beta(f(a))$$

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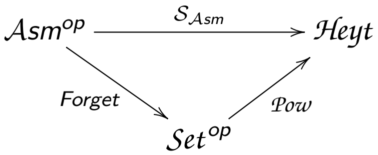
$\mathcal{PA}sm \subseteq_{full} \mathcal{A}sm$ on those (A, α) where $\alpha(a)$ is a singleton, i.e.

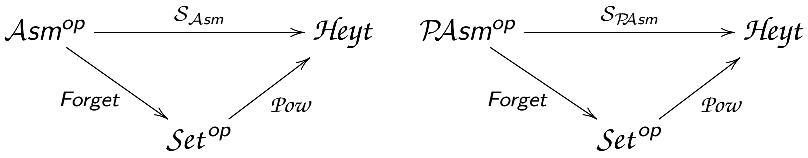
$$\alpha \colon A \to \mathbb{N}$$

Application to assemblies

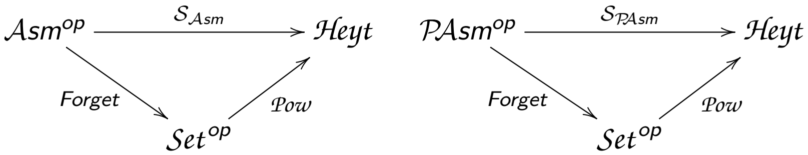
Application to assemblies



Application to assemblies



Application to assemblies



Theorem: $\widehat{\mathcal{S}_{\mathcal{P}\mathcal{A}sm}} \equiv \mathcal{S}_{\mathcal{A}sm}$

Application to assemblies (TCT), (CT)

$$\mathcal{S}_{\mathcal{PA}sm} \colon \mathcal{PA}sm^{op} \longrightarrow \mathcal{H}eyt \qquad \mathbf{N} = (\mathbb{N}, \mathrm{id}_{\mathbb{N}})$$

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- (CT) fails to hold

  $$\forall_{x:\mathbf{N}} \exists_{y:\mathbf{N}} R(x,y) \to \exists_{e:\mathbf{N}} \forall_{x:\mathbf{N}} \exists_{y:\mathbf{N}} \left[\mathsf{T}(e,x,y) = 1 \wedge R(x, \mathsf{U}(y))\right]$$

  take for R the graph of a non-computable function

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  take for R the graph of a non-computable function

- (TCT) holds

  $$\forall_{f:\mathbf{N}^{\mathbf{N}}} \exists_{e:\mathbf{N}} \forall_{x:\mathbf{N}} \exists_{y:\mathbf{N}} \; [\mathrm{T}(e,x,y) = 1 \land \mathrm{U}(y) = f(x)]$$

  definition of arrow in $\mathcal{PA}sm$

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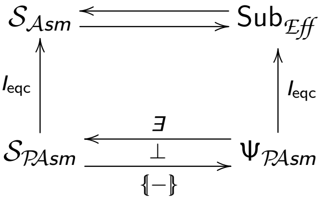
  definition of arrow in $\mathcal{PA}sm$ + Skolem arrows

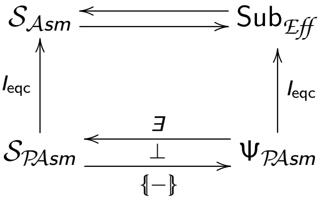
Application to assemblies (TCT), (CT)



Application to assemblies (TCT), (CT)

(TCT)

Skolem arrows



Application to assemblies (TCT), (CT)

$$\begin{array}{ccccc}
(TCT) & \mathcal{S}_{\mathcal{A}sm} & \rightleftarrows & \mathrm{Sub}_{\mathcal{E}\!f\!f} & (CT) \\
 & \uparrow I_{\mathrm{eqc}} & & \uparrow I_{\mathrm{eqc}} & \\
(TCT) & \mathcal{S}_{\mathcal{PA}sm} & \underset{\{-\}}{\overset{\exists}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} & \Psi_{\mathcal{PA}sm} & (CT) \\
\text{Skolem arrows} & & & &
\end{array}$$

Thank you

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