# An axiomatic approach to Gabriel-Ulmer duality

Ivan Di Liberti June 5, 2018

## Structure of the talk

• Give a definition of accessible and (locally) presentable object in a 2-category.

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- Cast a Gabriel-Ulmer duality for this definition.

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- S is a KZ-monad over K.
- P is a KZ-monad which is also a Yoneda structure (over K).
- $\bullet~Y$  is representably fully faithful + something else.

## Definition

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#### Remark

This does not imply that P(G) is accessible.

## **Representation Theorem**

The following are equivalent:

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Can we cast a Gabriel Ulmer duality for such a weak notion of accessibity? Well, maybe one should start by recalling what Gabriel Ulmer duality is.

There is an equivalence of categories.

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### Definition

Given a context  $S\overset{Y}{\Rightarrow}P$  a Gabriel Ulmer envelope  $\widehat{(\ \_)}$  for Y is an addition KZ monad such that

$$S(\widehat{(-)}) \cong P(-)$$

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# **Gabriel Ulmer Duality**

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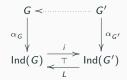
- $\widehat{(-)}$  is soaking;
- S is climbable;

then

$$\mathsf{Alg}(\widehat{(\ _{ ext{-}})})^{\mathsf{op}}\cong\mathsf{Pres}(Y).$$

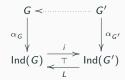
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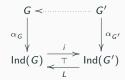


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