#### Generalized symmetries and arithmetic applications

James Borger

Australian National University

Category Theory 2018 University of the Azores Ponta Delgada, 2018/07/12

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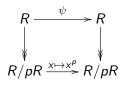
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  - ► Witt vectors and A-rings are important in arithmetic algebraic geometry
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- Today: open questions, the work of other people, some of my own

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R=ring (commutative, with 1)

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Frobenius lift

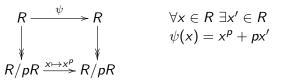


$$\forall x \in R \exists x' \in R \\ \psi(x) = x^p + px'$$

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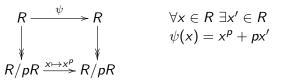


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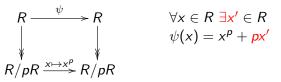


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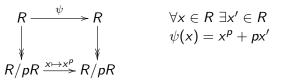
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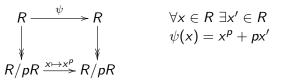
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- Property of existence  $\rightarrow$  a structure

A *p*-derivation on *R* is a function  $\delta \colon R \to R$  modeled on

$$\delta(x) = x' = \frac{\psi(x) - x^{p}}{p},$$

i.e., satisfying all the axioms it does when  $\psi$  is a Frobenius lift and  ${\it R}$  is  ${\it p}\mbox{-torsion}$  free:

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$$\delta(x+y) = \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{1}{p} {p \choose i} x^i y^{p-i}$$
$$\delta(xy) = \delta(x)y^p + x^p \delta(y) + p\delta(x)\delta(y)$$
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 $\{p$ -derivations on  $R\} \rightarrow \{Frobenius \text{ lifts on } R\}$ 

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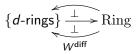
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 $\{p\text{-derivations on } R\} \xrightarrow{\sim} \{\text{Frobenius lifts on } R\}, \text{ if } R \text{ is } p\text{-tor-free}$ 

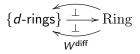
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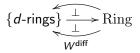


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$$W^{\mathrm{diff}}(R) = \left\{ \sum_{n} a_n \frac{t^n}{n!} \mid a_n \in R \right\}, \quad d = d/dt$$

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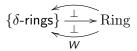


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$$= \left\{ (a_0, a_1, \dots) \right\}, \quad d = \text{shift}$$

Multiplication law at the *n*-th component is given by the Leibniz rule for  $d^{\circ n}(xy)$ :

$$(a_0,\ldots) \times (b_0,\ldots) = (a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + 2a_1b_1 + a_2b_0,\ldots)$$

 $\{\delta\text{-rings}\} \longrightarrow \operatorname{Ring}$ 



$$\{\delta\text{-rings}\} \xrightarrow[W]{\perp} \operatorname{Ring}$$

$$W(R) = R \times R \times R \times \cdots, \quad \delta(a_0, a_1, \dots) = (a_1, a_2, \dots)$$

Mulitiplication at the *n*-th component is again given by the Leibniz rule for  $\delta^{\circ n}(xy)$ , but now the same is true for addition!

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- Witt vectors are a machine for functorially lifting rings from characteristic p to characteristic 0
- Better: Witt vectors are a machine for adding a Frobenius lift to your ring, interpreted in an intelligent way

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 de Rham cohomology has problems in characteristic p: any function f<sup>p</sup> is a closed 0-form

$$d(f^p) = pf^{p-1} df = 0$$

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 Thus, if one is sufficiently enlightened, the concept of Frobenius lift, or *p*-derivation, leads automatically to crystalline cohomology.

(Tall–Wraith, Bergman–Hausknecht, Wieland & me, Stacey–Whitehouse)

 $C = \!\! a$  category of 'algebras' (rings, groups, Lie algebras,  $\ldots$  )

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A composition object of C is an object P of C plus a comonad structure on the functor it represents. ('Tall–Wraith monad object')

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  - In fact, the composition ring Z[e, δ, δ<sup>°2</sup>,...] cannot be generated by linear operators! It is fundamentally nonlinear.

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- ▶ Is it possible to classify all composition objects in Ring?
  - ► Carlson: Yes, if we allow denominators
  - Buium: Some positive classification results for composition rings generated by a single operator
  - ► All known examples come from linear operators or lifting Frobenius-like constructions from char *p* to char 0.

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## Imperative task #2 (with Garner)

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- ▶ END( $\mathbb{Z}$ )  $\stackrel{?}{=}$  {quasi-polynomials  $\mathbb{Z} \to \mathbb{Z}$ } (with Garner)
- ► END( $\mathbb{F}_p[t]$ ) = ?. Includes derivation d/dt, t-derivation  $f \mapsto (f - f^q)/t$ ,...

Principal categories of algebraic geometry:

 $\operatorname{Ring}^{\operatorname{op}} = \operatorname{Aff} \subset \operatorname{Sch} \subset \operatorname{AlgSp} \subset \operatorname{Sh}_{\operatorname{\acute{e}t}}(\operatorname{Aff}) \subset \operatorname{PSh}(\operatorname{Aff})$ 

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- Monoid and Lie algebra actions (linear symmetries) are OK: G-schemes, g-schemes
- Can this be done for *p*-derivations and similar non-linear symmetries? (Yes! See below.)
- Can this be done for *every* composition ring?
- Could there some kind of new generalized symmetry structures that exist only at the non-affine level?

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 $W_{n*}(X)$ :  $C \mapsto X(W_n(C))$ ,

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- ► This allows us to extend the theory of *p*-derivations,  $\delta$ -structures, and Witt vectors from rings to schemes  $\rightarrow$ " $\delta$ -equivariant algebraic geometry"
- ► The proof (Illusie, van der Kallen, Langer–Zink, me) is not formal!

Given a finite extension  $K/\mathbb{Q}$ , is there an explicit description of  $K^{ab}$ , its maximal Galois extension with abelian Galois group?

K = Q: Yes, the Kronecker–Weber theorem (1853–1896): adjoin all roots of unity exp(<sup>2πi</sup>/<sub>n</sub>) to Q

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- New idea: Use periodic points on  $\Lambda_{\mathcal{K}}$ -schemes instead!

Fix a finite extension  $K/\mathbb{Q}$ . Let  $\mathcal{O}_K$  denote its subring of algebraic integers. Let R be an  $\mathcal{O}_K$ -algebra.

• A  $\Lambda_K$ -structure on R is a commuting family of endomorphisms  $\psi_{\mathfrak{p}}$ , one for each nonzero prime ideal  $\mathfrak{p} \subset \mathcal{O}_K$  such that  $\psi_{\mathfrak{p}}(x) \equiv x^{N(\mathfrak{p})} \mod \mathfrak{p}R$ , where  $N(\mathfrak{p}) = |\mathcal{O}_K/\mathfrak{p}|$ .

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- $\rightarrow$  composition  $\mathcal{O}_{\mathcal{K}}$ -algebra  $\Lambda_{\mathcal{K}}$ , again nonlinear!
- Wilkerson, Joyal:  $\Lambda_{\mathbb{Q}}$ -ring =  $\lambda$ -ring as in K-theory

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- ► Any answer, positive or negative, for any other *K* would be very interesting!

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- ► Can one classify the composition objects in CAlg<sub>ℝ>0</sub>?
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- There must be many examples of other categories of algebras with generalized symmetries which are interesting and important!