Monads and theories

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- Many generalisations of this story – other bases than **Set**, enrichment, other shapes of operations than finite ...
- Today - a general class of monad–theory correspondences, that arise naturally. Joint work with Richard Garner – see "Monads and theories"(BG18).
- Closely related to, and inspired by, the notions of monad and theories with arities of Berger, Mellies and Weber (BMW12) – but has advantages.

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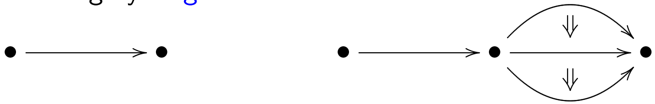

- Globular sets indexing operations in higher categories.

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Object: a pair $(X \in \mathcal{E}, F : \mathcal{T}^{op} \to \mathcal{V})$ with $N_K X = F \circ J^{op} : \mathcal{A}^{op} \to \mathcal{T}^{op} \to \mathcal{V}$. (See also Tom Avery's prototheories.)

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- The functor $\mathbf{Mod}_c(\mathcal{T}) \to \mathbf{Mod}(\mathcal{T})$ from concrete to non-concrete models is an equivalence.

From a monad to a pretheory

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- Gives a functor

$$R : \mathbf{Mnd}(\mathcal{E}) \to \mathbf{Preth}_{\mathcal{A}}(\mathcal{E}) : \mathrm{T} \mapsto J_T : \mathcal{A} \to \mathcal{A}_{\mathrm{T}}$$

from monads on $\mathcal{E}$ to $\mathcal{A}$-pretheories.

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- Gives a functor $L : \mathbf{Preth}_{\mathcal{A}}(\mathcal{E}) \to \mathbf{Mnd}(\mathcal{E})$.

The adjunction between monads and pretheories

- Theorem (BG18)

  The two constructions form an adjoint pair

$$\mathbf{Mnd}(\mathcal{E}) \underset{R}{\overset{L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{Preth}_{\mathcal{A}}(\mathcal{E}) \qquad .$$

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  - What are the fixpoints?

Fixpoints 1 - $\mathcal{A}$-nervous monads

- Recall

$$\mathcal{A} \xrightarrow{J_T} \mathcal{A}_T \xrightarrow{K_T} \mathcal{E}^T = \mathcal{A} \xrightarrow{K} \mathcal{E} \xrightarrow{F^T} \mathcal{E}^T$$

- Theorem (Weber's nerve theorem)

  If the monad T has arities $\mathcal{A}$ then

  1. *$K_T : \mathcal{A}_T \to \mathcal{E}^T$ is dense (i.e. $N_{K_T} : \mathcal{E}^T \to [\mathcal{A}_T^{op}, \mathcal{V}]$ is fully faithful) and*
  2. *$X : \mathcal{A}_T^{op} \to \mathcal{V}$ is a K_T-nerve iff $X \circ J_T^{op} : \mathcal{A}^{op} \to \mathcal{A}_T^{op} \to \mathcal{V}$ is a K-nerve.*

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  We say that a monad T is $\mathcal{A}$-nervous if Properties (1) and (2) above hold.

  Theorem (BG18)

  A monad T is $\mathcal{A}$-nervous if and only if $\epsilon_T : LRT \to T$ *is invertible.*

Fixpoints 2 - $\mathcal{A}$-theories

- A pretheory $J : \mathcal{A} \to \mathcal{T}$ is an $\mathcal{A}$-theory if for each $X \in \mathcal{T}$ the functor $\mathcal{T}(J-, X) : \mathcal{A}^{op} \to \mathcal{T}^{op} \to \mathcal{V}$ is a K-nerve.

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- Θ_0-theories are precisely the globular theories of Berger. They capture Batanin higher dimensional categories (Berger02). The Grothendieck weak ω-groupoids introduced by Maltsiniotis in 2010 are defined as models of certain globular theories – so we capture these.

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- Colimits of nervous monads are algebraic: sent to limits by $Alg : \mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})^{op} \to \mathbf{Cat}/\mathcal{E}$.
- $\mathbf{Sig}_{\mathcal{A}}(\mathcal{E}) = [obA, \mathcal{E}]$ the category of $\mathcal{A}$-signatures. E.g. in the classical case we get usual finitary signatures $[ob\mathbb{F}, \mathbf{Set}]$.

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  The monad $T = UF$ on $\mathbf{Sig}_{\mathcal{A}}(\mathcal{E})$ has $\mathbf{Mnd}_{\mathcal{A}}(\mathcal{E})$ as its category of algebras. In particular, the nervous monads are the colimit closure in $\mathbf{Mnd}(\mathcal{E})$ of the free monads on $\mathcal{A}$-signatures.

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  - Hence $\mathbb{F}$-nervous monads are the filtered colimit preserving ones, etc.
  - But Δ_0 and Θ_0 are not saturated – here we go beyond the classical setting.