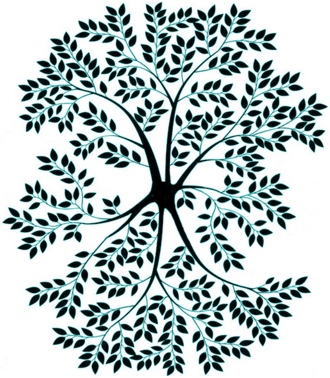
Categorified cyclic operads (in nature)

Category Theory 2018
University of Azores



Jovana Obradović

& Pierre-Louis Curien

Overview

1. Categorification
2. Cyclic operads
3. Categorified cyclic operads
   3.1 The coherence theorem
   3.2 Categorified cyclic operads “in nature”
   3.3 Polytopes of categorified cyclic operads

CATEGORIFICATION

Relevant examples of categorified structures

sets	categories
functions	functors
equalities	coherent isomorphisms

Relevant examples of categorified structures

sets	categories
functions	functors
equalities	coherent isomorphisms

Coherence of symmetric monoidal categories

S. Mac Lane
Categories for the Working Mathematician
Springer, 1997

$$\beta_{f,g,h} : (fg)h \to f(gh) \qquad \gamma_{f,g} : fg \to gf$$

Relevant examples of categorified structures

sets	categories
functions	functors
equalities	coherent isomorphisms

Coherence of symmetric monoidal categories

S. Mac Lane
Categories for the Working Mathematician
Springer, 1997

$$\beta_{f,g,h} : (fg)h \to f(gh) \qquad \gamma_{f,g} : fg \to gf$$

Coherence of categorified non-symmetric skeletal operads

K. Došen, Z. Petrić
Weak Cat-operads
Logical Methods in Computer Science, 2009

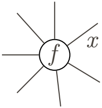
$$\beta^{i,j}_{f,g,h} : (f \circ_i g) \circ_j h \to f \circ_i (g \circ_{j-i+1} h) \qquad \theta^{i,j}_{f,g,h} : (f \circ_i g) \circ_j h \to (f \circ_j h) \circ_{i+n-1} g$$

CYCLIC OPERADS

Cyclic operads: definition

$$\mathcal{C} : \mathbf{Bij}^{op} \to \mathbf{Set}$$

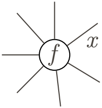
$\mathcal{C}(X) \ni$

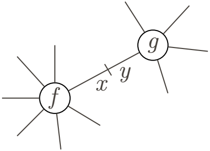


Cyclic operads: definition

$$\mathcal{C} : \mathbf{Bij}^{op} \to \mathbf{Set}$$

$$_{x}\circ_{y} : \mathcal{C}(X) \times \mathcal{C}(Y) \to \mathcal{C}(X\backslash\{x\} \cup Y\backslash\{y\})$$

$\mathcal{C}(X) \ni$ 

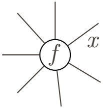


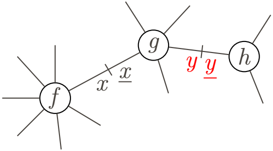
Cyclic operads: definition

$$\mathcal{C} : \mathbf{Bij}^{op} \to \mathbf{Set}$$

$$_{x}\circ_{y} : \mathcal{C}(X) \times \mathcal{C}(Y) \to \mathcal{C}(X \setminus \{x\} \cup Y \setminus \{y\})$$

$$(f\ _{x}\circ_{\underline{x}}\ g)\ _{y}\circ_{\underline{y}}\ h = f\ _{x}\circ_{\underline{x}}\ (g\ _{y}\circ_{\underline{y}}\ h) \qquad f\ _{x}\circ_{y}\ g = g\ _{y}\circ_{x}\ f$$

$\mathcal{C}(X) \ni$ 

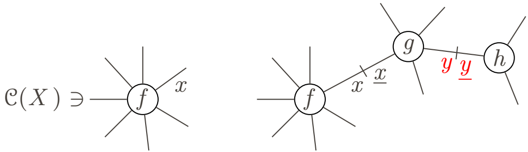


Cyclic operads: definition

$$\mathcal{C} : \mathbf{Bij}^{op} \to \mathbf{Set}$$

$$_x\circ_y : \mathcal{C}(X) \times \mathcal{C}(Y) \to \mathcal{C}(X\setminus\{x\} \cup Y\setminus\{y\})$$

$$(f\ {}_x\circ_{\underline{x}}\ g)\ {}_y\circ_{\underline{y}}\ h = f\ {}_x\circ_{\underline{x}}\ (g\ {}_y\circ_{\underline{y}}\ h) \qquad f\ {}_x\circ_y\ g = g\ {}_y\circ_x\ f$$



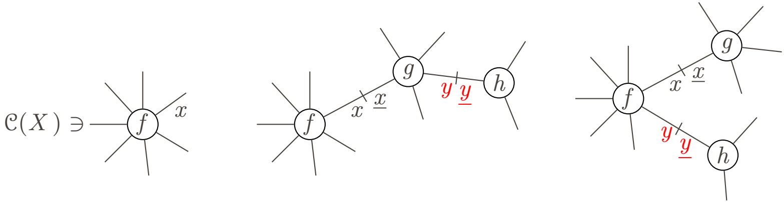
$$f^{\sigma_1}\ {}_{\sigma_1^{-1}(x)}\circ_{\sigma_2^{-1}(y)}\ g^{\sigma_2} = (f\ {}_x\circ_y\ g)^{\sigma}$$

Cyclic operads: definition

$$\mathcal{C} : \mathbf{Bij}^{op} \to \mathbf{Set}$$

$${}_x\circ_y : \mathcal{C}(X) \times \mathcal{C}(Y) \to \mathcal{C}(X\backslash\{x\} \cup Y\backslash\{y\})$$

$$(f\ {}_x\circ_{\underline{x}}\ g)\ {}_y\circ_{\underline{y}}\ h = f\ {}_x\circ_{\underline{x}}\ (g\ {}_y\circ_{\underline{y}}\ h) \qquad f\ {}_x\circ_y\ g = g\ {}_y\circ_x\ f$$



$$f^{\sigma_1}\ {}_{\sigma_1^{-1}(x)}\circ_{\sigma_2^{-1}(y)}\ g^{\sigma_2} = (f\ {}_x\circ_y\ g)^{\sigma}$$

Parallel associativity

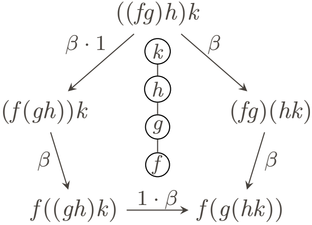
$$(f\ {}_x\circ_{\underline{x}}\ g)\ {}_y\circ_{\underline{y}}\ h = (g\ {}_{\underline{x}}\circ_x\ f)\ {}_y\circ_{\underline{y}}\ h = g\ {}_{\underline{x}}\circ_x\ (f\ {}_y\circ_{\underline{y}}\ h) = (f\ {}_y\circ_{\underline{y}}\ h)\ {}_x\circ_{\underline{x}}\ g$$

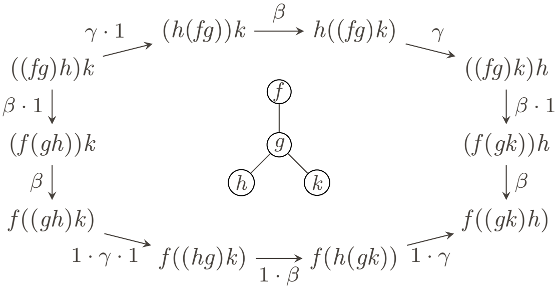
CATEGORIFIED CYCLIC OPERADS

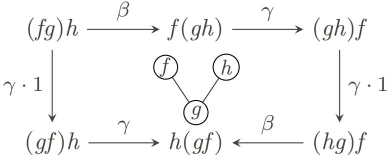
$$\mathcal{C} : \mathbf{Bij}^{op} \to \mathbf{Cat}$$

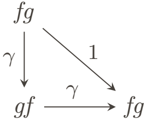
$$_{x}\circ_{y} : \mathcal{C}(X) \times \mathcal{C}(Y) \to \mathcal{C}(X\backslash\{x\} \cup Y\backslash\{y\})$$

$$\beta^{x,\underline{x};y,\underline{y}}_{f,g,h} : (f\ {}_{x}\circ_{\underline{x}}\ g)\ {}_{y}\circ_{\underline{y}}\, h \to f\ {}_{x}\circ_{\underline{x}}\ (g\ {}_{y}\circ_{\underline{y}}\, h) \qquad \gamma^{x,y}_{f,g} : f\ {}_{x}\circ_{y}\ g \to g\ {}_{y}\circ_{x}\ f$$









$$\beta^{\sigma}_{f,g,h} = \beta_{f^{\sigma_1}, g^{\sigma_2}, h^{\sigma_3}}$$

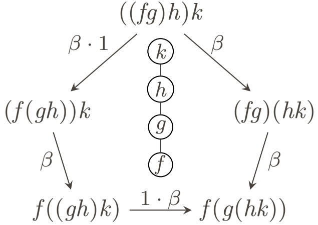
$$\gamma^{\sigma}_{f,g} = \gamma_{f^{\sigma_1}, g^{\sigma_2}}$$

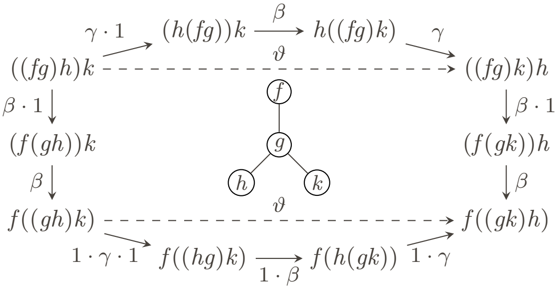
$$(\varphi \cdot \psi)^{\sigma} = \varphi^{\sigma_1} \cdot \psi^{\sigma_2}$$

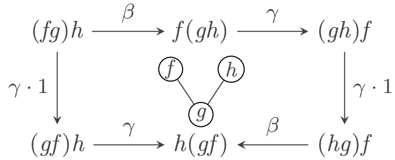
$$\mathcal{C} : \mathbf{Bij}^{op} \to \mathbf{Cat}$$

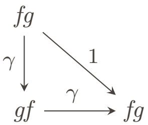
$$_{x}\circ_{y} : \mathcal{C}(X) \times \mathcal{C}(Y) \to \mathcal{C}(X \backslash \{x\} \cup Y \backslash \{y\})$$

$$\beta^{x,\underline{x};y,\underline{y}}_{f,g,h} : (f\ _{x}\circ_{\underline{x}}\ g)\ _{y}\circ_{\underline{y}}\ h \to f\ _{x}\circ_{\underline{x}}\ (g\ _{y}\circ_{\underline{y}}\ h) \qquad \gamma^{x,y}_{f,g} : f\ _{x}\circ_{y}\ g \to g\ _{y}\circ_{x}\ f$$









$$\beta^{\sigma}_{f,g,h} = \beta_{f^{\sigma_1},g^{\sigma_2},h^{\sigma_3}}$$

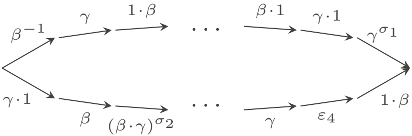
$$\gamma^{\sigma}_{f,g} = \gamma_{f^{\sigma_1},g^{\sigma_2}}$$

$$(\varphi \cdot \psi)^{\sigma} = \varphi^{\sigma_1} \cdot \psi^{\sigma_2}$$

The coherence theorem

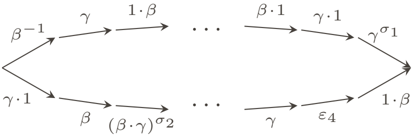
The coherence theorem: a formal language

Every diagram of canonical arrows in $\mathcal{C}(X)$ *commutes.*



The coherence theorem: a formal language

Every diagram of canonical arrows in $\mathcal{C}(X)$ commutes.



For the syntax $\texttt{Free}_{\underline{\mathcal{C}}}$ of $\beta\gamma\sigma$-diagrams & the coherence theorem:

P.-L. Curien, J. Obradović
Categorified cyclic operads
arXiv:1706.06788v2, 2018

The proof scheme

Coherence of categorified non-skeletal cyclic operads with symmetries

All $\beta\gamma\sigma$-diagrams in $\mathcal{C}(X)$ commute.

Došen & Petrić: Coherence of categorified skeletal operads without symmetries

For an arbitrary categorified operad $\mathcal{O}$,
all $\beta\theta$-diagrams in $\mathcal{O}(n)$ commute.

The proof scheme

Coherence of categorified non-skeletal cyclic operads with symmetries

All $\beta\gamma\sigma$-diagrams in $\mathcal{C}(X)$ commute.

1st reduction:
removing symmetries

All $\beta\gamma$-diagrams in $\mathcal{C}(X)$ commute.

Došen & Petrić: Coherence of categorified skeletal operads without symmetries

For an arbitrary categorified operad $\mathcal{O}$,
all $\beta\theta$-diagrams in $\mathcal{O}(n)$ commute.

The proof scheme

Coherence of categorified non-skeletal cyclic operads with symmetries

All $\beta\gamma\sigma$-diagrams in $\mathcal{C}(X)$ commute.

↓ 1st reduction: removing symmetries

All $\beta\gamma$-diagrams in $\mathcal{C}(X)$ commute.

↓ 2nd reduction: removing cyclicity

All $\beta\vartheta$-diagrams in $\mathcal{C}(X)$ commute.

Došen & Petrić: Coherence of categorified skeletal operads without symmetries

For an arbitrary categorified operad $\mathcal{O}$, all $\beta\theta$-diagrams in $\mathcal{O}(n)$ commute.

The proof scheme

Coherence of categorified non-skeletal cyclic operads with symmetries

All $\beta\gamma\sigma$-diagrams in $\mathcal{C}(X)$ commute.

↓ 1st reduction: removing symmetries

All $\beta\gamma$-diagrams in $\mathcal{C}(X)$ commute.

↓ 2nd reduction: removing cyclicity

All $\beta\vartheta$-diagrams in $\mathcal{C}(X)$ commute.

↓ 3rd reduction: skeletalisation

Došen & Petrić: Coherence of categorified skeletal operads without symmetries

For an arbitrary categorified operad $\mathcal{O}$, all $\beta\theta$-diagrams in $\mathcal{O}(n)$ commute.

The proof scheme

Coherence of categorified non-skeletal cyclic operads with symmetries

All $\beta\gamma\sigma$-diagrams in $\mathcal{C}(X)$ commute.

↓

1st reduction:
removing symmetries

$$(f\ {}_x\circ_y\ g)^\sigma \rightsquigarrow f^{\sigma_1}\ {}_{\sigma_1^{-1}(x)}\circ_{\sigma_2^{-1}(y)}\ g^{\sigma_2}$$

↓

All $\beta\gamma$-diagrams in $\mathcal{C}(X)$ commute.

↓

2nd reduction:
removing cyclicity

↓

All $\beta\vartheta$-diagrams in $\mathcal{C}(X)$ commute.

↓

3rd reduction:
skeletalisation

↓

Došen & Petrić: Coherence of categorified skeletal operads without symmetries

For an arbitrary categorified operad $\mathcal{O}$,
all $\beta\theta$-diagrams in $\mathcal{O}(n)$ commute.

The proof scheme

Coherence of categorified non-skeletal cyclic operads with symmetries

All $\beta\gamma\sigma$-diagrams in $\mathcal{C}(X)$ commute.

↓

1st reduction:
removing symmetries

$$(f\ {}_x\circ_y\ g)^{\sigma} \rightsquigarrow f^{\sigma_1}\ {}_{\sigma_1^{-1}(x)}\circ_{\sigma_2^{-1}(y)}\ g^{\sigma_2}$$

↓

All $\beta\gamma$-diagrams in $\mathcal{C}(X)$ commute.

↓

2nd reduction:
removing cyclicity

↓

All $\beta\vartheta$-diagrams in $\mathcal{C}(X)$ commute.

↓

3rd reduction:
skeletalisation

$$\mathbf{Bij} \simeq \Sigma$$

↓

Došen & Petrić: Coherence of categorified skeletal operads without symmetries

For an arbitrary categorified operad $\mathcal{O}$,
all $\beta\theta$-diagrams in $\mathcal{O}(n)$ commute.

2nd reduction: removing cyclicity

$$\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X) \ni (\underline{a}\ {}_x\Box_{\underline{x}}\ \underline{b})\ {}_y\Box_{\underline{y}}\ \underline{c} \quad \Leftrightarrow \quad ($$

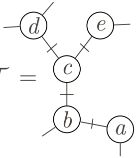
$$, (\underline{a}\,\underline{b})\underline{c}) \in \underline{\mathtt{T}}_{\underline{\mathcal{C}}}(X)$$

2nd reduction: removing cyclicity

$$\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X) \ni (\underline{a}\ {}_x\square_{\underline{x}}\ \underline{b})\ {}_y\square_{\underline{y}}\ \underline{c} \quad \Leftrightarrow \quad \Big($$

$$,\ (\underline{a}\,\underline{b})\underline{c}\Big) \in \underline{\mathtt{T}}_{\underline{\mathcal{C}}}(X)$$

$\mathcal{T} =$



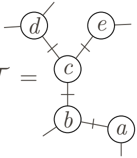
$$a(b\,(d\,(ec)))$$

2nd reduction: removing cyclicity

$$\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X) \ni (\underline{a}\ {}_x\square_{\underline{x}}\ \underline{b})\ {}_y\square_{\underline{y}}\ \underline{c} \quad \Leftrightarrow \quad \Big($$

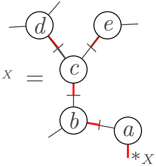
$$,\ (\underline{a}\,\underline{b})\underline{c}\Big) \in \underline{\mathtt{T}}_{\underline{\mathcal{C}}}(X)$$

$\mathcal{T} =$



$a(b\,(d\,(ec)))$

$\mathcal{T}^{\cdot * X} =$

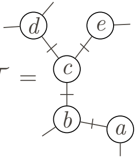


2nd reduction: removing cyclicity

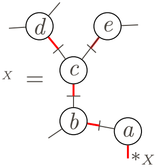
$\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X) \ni (\underline{a}\ {}_x\square_{\underline{x}}\ \underline{b})\ {}_y\square_{\underline{y}}\ \underline{c} \quad \Leftrightarrow \quad ($

$, (\underline{a}\,\underline{b})\underline{c}) \in \underline{\mathtt{T}}_{\underline{\mathcal{C}}}(X)$

$\mathcal{T} =$



$\mathcal{T}^{\cdot *_X} =$



$$a(b\,(d\,(ec)))$$
$$\big\downarrow\ 1_a \cdot 1_b \cdot \gamma_{d,ec}$$
$$a(b\,((ec)\,d))$$
$$\big\downarrow\ 1_a \cdot 1_b \cdot \gamma_{e,c} \cdot 1_d$$
$$a(b\,((ce)\,d))$$

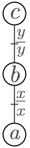
2nd reduction: removing cyclicity

$$\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X) \ni (\underline{a}\ {}_x\square_{\underline{x}}\ \underline{b})\ {}_y\square_{\underline{y}}\ \underline{c} \quad \Leftrightarrow \quad ($$

$$, (\underline{a}\,\underline{b})\underline{c}) \in \underline{\mathtt{T}}_{\underline{\mathcal{C}}}(X)$$

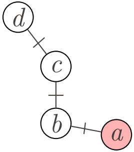
2nd reduction: removing cyclicity

$$\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X) \ni (\underline{a}\ {}_{x}\square_{\underline{x}}\ \underline{b})\ {}_{y}\square_{\underline{y}}\ \underline{c} \quad \Leftrightarrow \quad \Big($$



$$, (\underline{a}\,\underline{b})\underline{c}\Big) \in \underline{\mathtt{T}}_{\underline{\mathcal{C}}}(X)$$

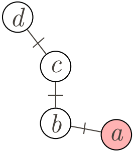
$$((ba)c)d \xrightarrow{\beta_{ba,c,d}} (ba)(cd)$$



$$\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X) \ni (\underline{a}\ {}_{x}\square_{\underline{x}}\ \underline{b})\ {}_{y}\square_{\underline{y}}\ \underline{c} \quad \Leftrightarrow \quad ($$

$$, (\underline{a}\,\underline{b})\underline{c}) \in \underline{\mathtt{T}}_{\underline{\mathcal{C}}}(X)$$

$$\begin{array}{ccc} ((ba)c)d & \xrightarrow{\beta_{ba,c,d}} & (ba)(cd) \\ \downarrow{\scriptstyle \gamma_{b,a}\cdot 1_c \cdot 1_d} & & \downarrow{\scriptstyle \gamma_{b,a}\cdot 1_{cd}} \\ ((ab)c)d & & (ab)(cd) \end{array}$$

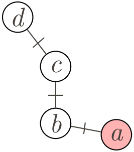


2nd reduction: removing cyclicity

$$\underline{\mathtt{Free}}_{\underline{\mathfrak{C}}}(X) \ni (\underline{a}\ {}_{x}\square_{\underline{x}}\ \underline{b})\ {}_{y}\square_{\underline{y}}\ \underline{c} \quad \Leftrightarrow \quad \Big($$

$$, (\underline{a}\,\underline{b})\underline{c}\Big) \in \underline{\mathtt{T}}_{\underline{\mathfrak{C}}}(X)$$

$$\begin{array}{ccc} ((ba)c)d & \xrightarrow{\beta_{ba,c,d}} & (ba)(cd) \\ {\scriptstyle \gamma_{b,a}\cdot 1_c \cdot 1_d}\downarrow & & \downarrow{\scriptstyle \gamma_{b,a}\cdot 1_{cd}} \\ ((ab)c)d & \xrightarrow{\beta_{ab,c,d}} & (ab)(cd) \end{array}$$

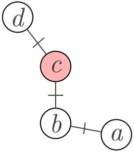


2nd reduction: removing cyclicity

$$\underline{\texttt{Free}}_{\underline{\mathcal{C}}}(X) \ni (\underline{a}\ {}_{x}\square_{\underline{x}}\ \underline{b})\ {}_{y}\square_{\underline{y}}\ \underline{c} \quad \Leftrightarrow \quad \Big($$

$$,\ (\underline{a}\,\underline{b})\underline{c}\Big) \in \underline{\texttt{T}}_{\underline{\mathcal{C}}}(X)$$

$$((ba)c)d \xrightarrow{\beta_{ba,c,d}} (ba)(cd)$$

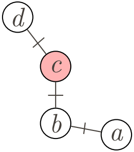


2nd reduction: removing cyclicity

$$\underline{\mathtt{Free}_{\mathcal{C}}}(X) \ni (\underline{a}\ {}_{x}\square_{\underline{x}}\ \underline{b})\ {}_{y}\square_{\underline{y}}\ \underline{c} \quad \Leftrightarrow \quad \Big($$

$$, (\underline{a}\,\underline{b})\underline{c}\Big) \in \underline{\mathtt{T}_{\mathcal{C}}}(X)$$

$$\begin{array}{ccc} ((ba)c)d & \xrightarrow{\beta_{ba,c,d}} & (ba)(cd) \\ \downarrow{\scriptstyle \gamma_{ba,c}\cdot 1_d} & & \downarrow{\scriptstyle \gamma_{ba,cd}} \\ (c(ba))d & & (cd)(ba) \end{array}$$

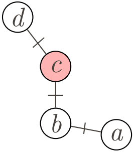


2nd reduction: removing cyclicity

$$\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X) \ni (\underline{a}\ {}_{x}\square_{\underline{x}}\ \underline{b})\ {}_{y}\square_{\underline{y}}\ \underline{c} \quad \Leftrightarrow \quad \Big($$

$$,\ (\underline{a}\,\underline{b})\underline{c}\Big) \in \underline{\mathtt{T}}_{\underline{\mathcal{C}}}(X)$$

$$\begin{array}{ccc}
((ba)c)d & \xrightarrow{\beta_{ba,c,d}} & (ba)(cd) \\
\gamma_{ba,c} \cdot 1_d \downarrow & & \downarrow \gamma_{ba,cd} \\
(c(ba))d & \xrightarrow{\vartheta_{c,ba,d}} & (cd)(ba)
\end{array}$$

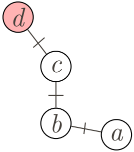


2nd reduction: removing cyclicity

$$\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X) \ni (\underline{a}\ {}_x\square_{\underline{x}}\ \underline{b})\ {}_y\square_{\underline{y}}\ \underline{c} \quad \Leftrightarrow \quad \Big($$

$$,\ (\underline{a}\,\underline{b})\underline{c}\Big) \in \underline{\mathtt{T}}_{\underline{\mathcal{C}}}(X)$$

$$((ba)c)d \xrightarrow{\beta_{ba,c,d}} (ba)(cd)$$

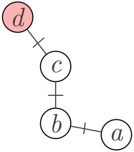


2nd reduction: removing cyclicity

$$\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X) \ni (\underline{a}\ {}_{x}\square_{\underline{x}}\ \underline{b})\ {}_{y}\square_{\underline{y}}\ \underline{c} \quad \Leftrightarrow \quad ($$

$$, (\underline{a}\,\underline{b})\underline{c}) \in \underline{\mathtt{T}}_{\underline{\mathcal{C}}}(X)$$

$$\begin{array}{ccc} ((ba)c)d & \xrightarrow{\beta_{ba,c,d}} & (ba)(cd) \\ \downarrow{\scriptstyle \gamma_{(ba)c,d}} & & \downarrow{\scriptstyle \gamma_{ba,cd}} \\ d((ba)c) & & (cd)(ba) \end{array}$$

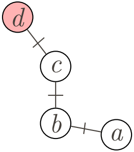


2nd reduction: removing cyclicity

$$\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X) \ni (\underline{a}\ {}_{x}\square_{\underline{x}}\ \underline{b})\ {}_{y}\square_{\underline{y}}\ \underline{c} \quad \Leftrightarrow \quad ($$

$$, (\underline{a}\,\underline{b})\underline{c}) \in \underline{\mathtt{T}}_{\underline{\mathcal{C}}}(X)$$

$$\begin{array}{ccc}
((ba)c)d & \xrightarrow{\beta_{ba,c,d}} & (ba)(cd) \\
\downarrow{\scriptstyle \gamma_{(ba)c,d}} & & \downarrow{\scriptstyle \gamma_{ba,cd}} \\
d((ba)c) & & (cd)(ba) \\
\downarrow{\scriptstyle 1_d \cdot \gamma_{ba,c}} & & \downarrow{\scriptstyle \gamma_{c,d} \cdot 1_{ba}} \\
d(c(ba)) & & (dc)(ba)
\end{array}$$

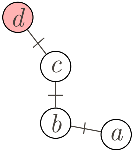


2nd reduction: removing cyclicity

$$\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X) \ni (\underline{a}\ {}_x\square_{\underline{x}}\ \underline{b})\ {}_y\square_{\underline{y}}\ \underline{c} \quad \Leftrightarrow \quad ($$

$$,\ (\underline{a}\,\underline{b})\underline{c}) \in \underline{\mathtt{T}}_{\underline{\mathcal{C}}}(X)$$

$$\begin{array}{ccc} ((ba)c)d & \xrightarrow{\beta_{ba,c,d}} & (ba)(cd) \\ {\scriptstyle \gamma_{(ba)c,d}}\downarrow & & \downarrow{\scriptstyle \gamma_{ba,cd}} \\ d((ba)c) & & (cd)(ba) \\ {\scriptstyle 1_d \cdot \gamma_{ba,c}}\downarrow & & \downarrow{\scriptstyle \gamma_{c,d} \cdot 1_{ba}} \\ d(c(ba)) & \xrightarrow{\beta^{-1}_{d,g,ca}} & (dc)(ba) \end{array}$$

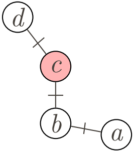


2nd reduction: removing cyclicity

$$\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X) \ni (\underline{a}\ {}_x\square_{\underline{x}}\ \underline{b})\ {}_y\square_{\underline{y}}\ \underline{c} \quad \Leftrightarrow \quad ($$

$$, (\underline{a}\,\underline{b})\underline{c}) \in \underline{\mathtt{T}}_{\underline{\mathcal{C}}}(X)$$

$$(ba)(cd) \xrightarrow{\gamma_{ba,cd}} (cd)(ba)$$

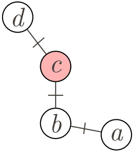


2nd reduction: removing cyclicity

$$\underline{\texttt{Free}}_{\underline{\mathcal{C}}}(X) \ni (\underline{a}\ {}_{x}\square_{\underline{x}}\ \underline{b})\ {}_{y}\square_{\underline{y}}\ \underline{c} \quad \Leftrightarrow \quad ($$

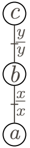
$$, (\underline{a}\,\underline{b})\underline{c}) \in \underline{\texttt{T}}_{\underline{\mathcal{C}}}(X)$$

$$\begin{array}{ccc} (ba)(cd) & \xrightarrow{\gamma_{ba,cd}} & (cd)(ba) \\ \downarrow{\scriptstyle \gamma_{ba,cd}} & & \downarrow{\scriptstyle 1_{ba,cd}} \\ (cd)(ba) & & (cd)(ba) \end{array}$$



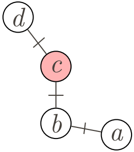
2nd reduction: removing cyclicity

$$\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X) \ni (\underline{a}\ {}_x\square_{\underline{x}}\ \underline{b})\ {}_y\square_{\underline{y}}\ \underline{c} \quad \Leftrightarrow \quad ($$



$$, (\underline{a}\,\underline{b})\underline{c}) \in \underline{\mathtt{T}}_{\underline{\mathcal{C}}}(X)$$

$$\begin{array}{ccc} (ba)(cd) & \xrightarrow{\gamma_{ba,cd}} & (cd)(ba) \\ \downarrow{\scriptstyle \gamma_{ba,cd}} & & \downarrow{\scriptstyle 1_{ba,cd}} \\ (cd)(ba) & \xrightarrow{1_{ba,cd}} & (cd)(ba) \end{array}$$



Categorified cyclic operads “in nature”

J. Bénabou
Les distributeurs
Université Catholique de Louvain
Institut de Mathématique Pure et Appliquée, rapport 33, 1973

Bicategory **Prof**

- objects: small categories
- 1-morphisms: profunctors $F : \mathbf{C} \nrightarrow \mathbf{D}$, i.e. functors $F : \mathbf{D}^{op} \times \mathbf{C} \to \mathbf{Set}$

$$G \circ F = \int^{d} F(d,-) \times G(-,d)$$

- 2-morphisms: natural transformations

Generalised profunctors as categorified cyclic operads

- $\mathbf{D}$ small category, $(-)^* : \mathbf{D}^{op} \to \mathbf{D}$ duality

$$F : \mathbf{D}^{op} \times \mathbf{D} \to \mathbf{Set} \quad \text{can be considered as} \quad F : \mathbf{D} \times \mathbf{D} \to \mathbf{Set}$$

Generalised profunctors as categorified cyclic operads

- $\mathbf{D}$ small category, $(-)^* : \mathbf{D}^{op} \to \mathbf{D}$ duality

$$F : \mathbf{D}^{op} \times \mathbf{D} \to \mathbf{Set} \quad \text{can be considered as} \quad F : \mathbf{D} \times \mathbf{D} \to \mathbf{Set}$$

- $\mathbf{D}^n$-profunctors: $\mathcal{C}(n) = [\mathbf{D}^n, \mathbf{Set}]$

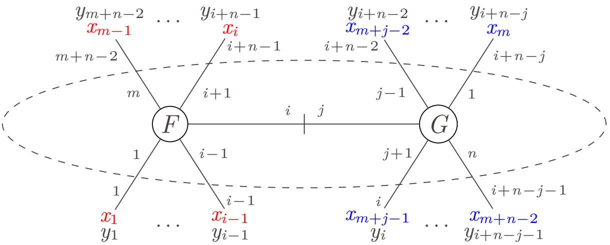
Generalised profunctors as categorified cyclic operads

- $\mathbf{D}$ small category, $(-)^* : \mathbf{D}^{op} \to \mathbf{D}$ duality

$$F : \mathbf{D}^{op} \times \mathbf{D} \to \mathbf{Set} \quad \text{can be considered as} \quad F : \mathbf{D} \times \mathbf{D} \to \mathbf{Set}$$

- $\mathbf{D}^n$-profunctors: $\mathcal{C}(n) = [\mathbf{D}^n, \mathbf{Set}]$
- ${}_i\circ_j : \mathcal{C}(m) \times \mathcal{C}(n) \to \mathcal{C}(m+n-2)$

$$(F \; {}_i\circ_j \; G)(y_1, \ldots, y_{m+n-2}) =$$
$$\int^{u,v} F(x_1, \ldots, x_{i-1}, u, x_i, \ldots, x_{m-1}) \times G(x_m, \ldots, x_{m+j-2}, v, x_{m+j-1}, \ldots, x_{m+n-2}) \times \mathbf{D}[u, v^*]$$



Feynman category for cyclic operads admits an odd version

R. Kaufmann, B. Ward
Feynman Categories
Astérisque (Société Mathématique de France), Numéro 387, 2017

Cyclic operads are representations of the Feynman category $\mathcal{C}yc$.

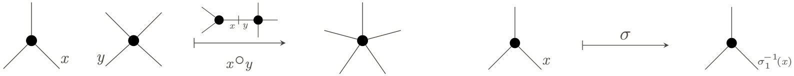
Feynman category for cyclic operads admits an odd version

R. Kaufmann, B. Ward
Feynman Categories
Astérisque (Société Mathématique de France), Numéro 387, 2017

Cyclic operads are representations of the Feynman category Cyc.

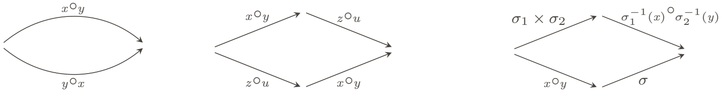
Generators-and-relations representation of Cyc:

- objects: sets of corollas
- morphisms:



- relations:

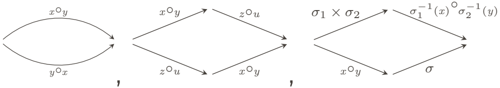
$$x \circ y = y \circ x \qquad x \circ y \, z \circ u = z \circ u \, x \circ y \qquad \sigma_1^{-1}(x) \circ \sigma_2^{-1}(y) (\sigma_1 \times \sigma_2) = \sigma \, x \circ y$$



Feynman category for cyclic operads admits an odd version

Cyc admits an **ordered presentation** if there exists

$sgn : \{$



$\} \to \{+, -\}$

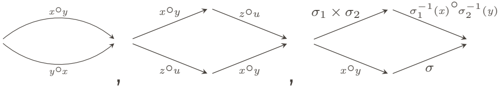
such that the **sign coherence** holds:

sgn extends multiplicatively to every "polygon";
every two parallel polygons receive the same sign

Feynman category for cyclic operads admits an odd version

Cyc admits an **ordered presentation** if there exists

$sgn : \{$



$\} \to \{+, -\}$

such that the **sign coherence** holds:

sgn extends multiplicatively to every "polygon";
every two parallel polygons receive the same sign

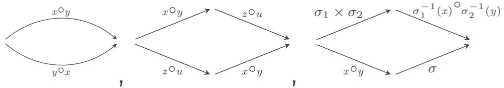
Theorem

Cyc can be ordered by assigning $-$ to ${}_x\circ_y = {}_y\circ_x$ and $+$ to other relations.

Feynman category for cyclic operads admits an odd version

Cyc admits an **ordered presentation** if there exists

$sgn :$ $\Bigg\{$



$\Bigg\} \to \{+, -\}$

such that the **sign coherence** holds:

sgn extends multiplicatively to every "polygon";
every two parallel polygons receive the same sign

Theorem

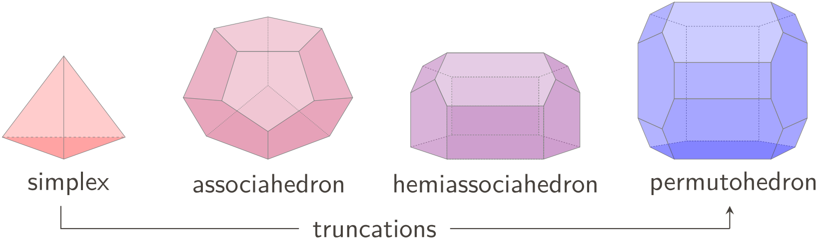
Cyc can be ordered by assigning $-$ to ${}_x\circ_y = {}_y\circ_x$ and $+$ to other relations.

$(Cyc, sgn) \quad \Rightarrow \quad ||(Cyc, sgn)||^{odd}$ $\mathbf{Ab}$-enriched Feynman category
for **anti-cyclic operads**

Polytopes of categorified cyclic operads

Categorified operads: Hypergraph polytopes

K. Došen, Z. Petrić
Hypergraph polytopes
Topology and its Applications 158, pp. 1405–1444, 2011



"the more hyperedges, the more truncations"

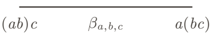
Hypergraph arrangements of hypercubes

Categorified operads	**Categorified cyclic operads**
associahedron K_n	K_n-arrangement of hypercubes
hemiassociahedron H_n	H_n-arrangement of hypercubes
permutohedron P_n	P_n-arrangement of hypercubes
$\cap\mid$	$\cap\mid$
hypergraph polytopes	hypergraph arrangements of hypercubes

Hypergraph arrangements of hypercubes

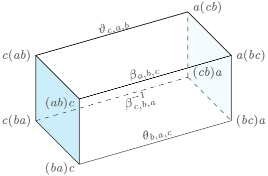
Categorified operads

associahedron K_n
hemiassociahedron H_n
permutohedron P_n
$\cap|$
hypergraph polytopes



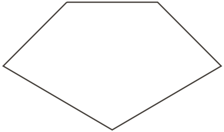
Categorified cyclic operads

K_n-arrangement of hypercubes
H_n-arrangement of hypercubes
P_n-arrangement of hypercubes
$\cap|$
hypergraph arrangements of hypercubes



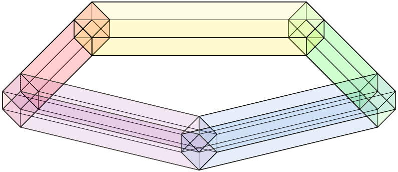
Categorified operads

associahedron K_n
hemiassociahedron H_n
permutohedron P_n
$\cap|$
hypergraph polytopes



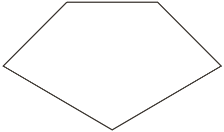
Categorified cyclic operads

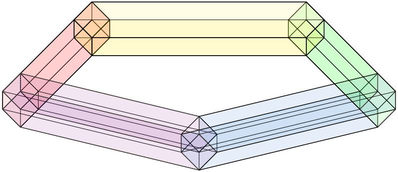
K_n-arrangement of hypercubes
H_n-arrangement of hypercubes
P_n-arrangement of hypercubes
$\cap|$
hypergraph arrangements of hypercubes



Hypergraph arrangements of hypercubes

Categorified operads	**Categorified cyclic operads**
associahedron K_n	K_n-arrangement of hypercubes
hemiassociahedron H_n	H_n-arrangement of hypercubes
permutohedron P_n	P_n-arrangement of hypercubes
∩|	∩|
hypergraph polytopes	hypergraph arrangements of hypercubes





$$\mathcal{A}_{ha}(\mathbf{H}) = \mathcal{A}(\mathbf{H}) \times \mathbf{Q}_{|H|}$$

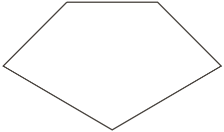
Hypergraph arrangements of hypercubes

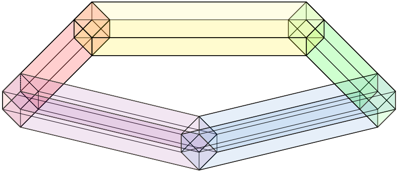
Categorified operads

associahedron K_n
hemiassociahedron H_n
permutohedron P_n
$\cap|$
hypergraph polytopes

Categorified cyclic operads

K_n-arrangement of hypercubes
H_n-arrangement of hypercubes
P_n-arrangement of hypercubes
$\cap|$
hypergraph arrangements of hypercubes





$$\mathcal{A}_{ha}(\mathbf{H}) = \mathcal{A}(\mathbf{H}) \times \mathbf{Q}_{|H|}$$

Thank you!