Galois theory, a logical path from Grothendieck's version to the fundamental theorem

Johan Felipe García Vargas

Departamento de Matemáticas
Universidad de los Andes

July 11, 2017

Outline

- Traditionally Galois theory is seen as a correspondence given by stabilizers and fixed points, Grothendieck frames it as a monadicity result.
- In model theory, internality implies pro-definable binding group and Galois correspondence. I will explain this from a categorical logic perspective as a natural consequence of a monadicity result.
- In this framework, Grothendieck's Galois theory is internality over finite sets and Tannakian duality can be immersed as internality over constructible sets.

Outline

- Traditionally Galois theory is seen as a correspondence given by stabilizers and fixed points, Grothendieck frames it as a monadicity result.
- In model theory, internality implies pro-definable binding group and Galois correspondence. I will explain this from a categorical logic perspective as a natural consequence of a monadicity result.
- In this framework, Grothendieck's Galois theory is internality over finite sets and Tannakian duality can be immersed as internality over constructible sets.

Outline

- Traditionally Galois theory is seen as a correspondence given by stabilizers and fixed points, Grothendieck frames it as a monadicity result.
- In model theory, internality implies pro-definable binding group and Galois correspondence. I will explain this from a categorical logic perspective as a natural consequence of a monadicity result.
- In this framework, Grothendieck's Galois theory is internality over finite sets and Tannakian duality can be immersed as internality over constructible sets.

Galois theory of fields

Fundamental theorem

Let $F \leq E$ be a Galois extension of fields, then the Galois group $G = \mathrm{Aut}(E/F)$ is pro-finite and there is a biyective correspondence between intermediate fields and closed subgroups of G.

$$\{K \mid F \leq K \leq E\} \underset{E^H}{\overset{\mathrm{Aut}(E/K)}{\rightleftarrows}} \{H \leq G \mid H \text{ is closed}\}$$

Grothendieck's version

Let $\omega : \mathcal{C} \to \mathcal{S}\mathrm{ets}_f$ be a *fundamental functor* from a *Galoisian* category, then $\pi = \mathrm{Aut}(\omega)$ is a pro-finite group and ω lifts to an equivalence between $\mathcal{C}$ and the category of finite continuous π-actions

Galois theory of fields

Fundamental theorem

Let $F \leq E$ be a Galois extension of fields, then the Galois group $G = \mathrm{Aut}(E/F)$ is pro-finite and there is a biyective correspondence between intermediate fields and closed subgroups of G.

Grothendieck's version

Let $\omega : \mathcal{C} \to \mathcal{S}\mathrm{ets}_f$ be a *fundamental functor* from a *Galoisian* category, then $\pi = \mathrm{Aut}(\omega)$ is a pro-finite group and ω lifts to an equivalence between $\mathcal{C}$ and the category of finite continuous π-actions $\mathcal{S}\mathrm{ets}_f^\pi$.

$$\begin{array}{ccc} \mathcal{C} & \longleftrightarrow & \mathcal{S}\mathrm{ets}_f^\pi \\ & \overset{\omega}{\searrow} & \downarrow \\ & & \mathcal{S}\mathrm{ets}_f \end{array}$$

Galois theory of fields

Fundamental theorem

Let $F \leq E$ be a Galois extension of fields, then the Galois group $G = \mathrm{Aut}(E/F)$ is pro-finite and there is a biyective correspondence between intermediate fields and closed subgroups of G.

Grothendieck's version

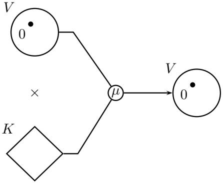
Let $\omega : \mathcal{C} \to \mathcal{S}\mathrm{ets}_f$ be a *fundamental functor* from a *Galoisian* category, then $\pi = \mathrm{Aut}(\omega)$ is a pro-finite group and ω lifts to an equivalence between $\mathcal{C}$ and the category of finite continuous π-actions $\mathcal{S}\mathrm{ets}_f^{\pi}$.

In particular, let $\mathcal{C}^{op}$ be the category of finite étale F-algebras split by E and take $\omega(X) = X(E) = \mathrm{Hom}_{F\text{-alg.}}(A, E)$ when $X = \mathrm{Spec}\, A$ in $\mathcal{C}$, then $\mathrm{Aut}(\omega) = \mathrm{Aut}(E/F)$ and $\mathcal{C}$ is equivalent to $\mathcal{S}\mathrm{ets}_f^{G}$.

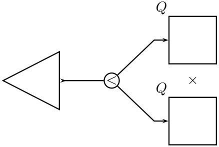
Categorical logic

Given a first order theory T

- The category of T-definables (including imaginary sorts) $\mathcal{T} = \mathrm{Def}(T^{eq})$ is a *boolean pre-topos*.
- Models are *logical functors* $\mathfrak{M} : \mathcal{T} \to \mathcal{S}\mathrm{ets}$.
- Interpretations are logical functors $\iota : \mathcal{T}_0 \to \mathcal{T}$.
- ι is an *immersion* if $\iota_X : \mathrm{Sub}_{\mathcal{T}_0}(X) \to \mathrm{Sub}_{\mathcal{T}}(\iota X)$ is an isomorphism for every X.



Function $\mu : K \times V \to V$.
Constant $0 \in V$

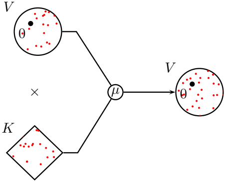


A binary relation $<$ in the sort Q.

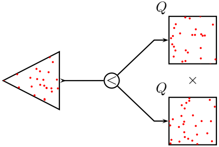
Categorical logic

Given a first order theory T

- The category of T-definables (including imaginary sorts) $\mathcal{T} = \mathrm{Def}(T^{eq})$ is a *boolean pre-topos*.
- Models are *logical functors* $\mathfrak{M} : \mathcal{T} \to \mathcal{S}\mathrm{ets}$.
- Interpretations are logical functors $\iota : \mathcal{T}_0 \to \mathcal{T}$.
- ι is an *immersion* if $\iota_X : \mathrm{Sub}_{\mathcal{T}_0}(X) \to \mathrm{Sub}_{\mathcal{T}}(\iota X)$ is an isomorphism for every X.



Function $\mu : K \times V \to V$.
Constant $0 \in V$

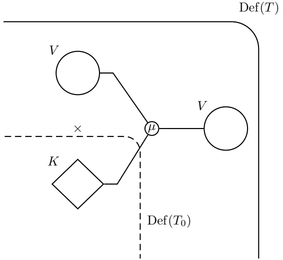


A binary relation $<$ in the sort Q.

Categorical logic

Given a first order theory T

- The category of T-definables (including imaginary sorts) $\mathcal{T} = \mathrm{Def}(T^{eq})$ is a *boolean pre-topos*.
- Models are *logical functors* $\mathfrak{M} : \mathcal{T} \to \mathcal{S}\mathrm{ets}$.
- Interpretations are logical functors $\iota : \mathcal{T}_0 \to \mathcal{T}$.
- ι is an *immersion* if $\iota_X : \mathrm{Sub}_{\mathcal{T}_0}(X) \to \mathrm{Sub}_{\mathcal{T}}(\iota X)$ is an isomorphism for every X.

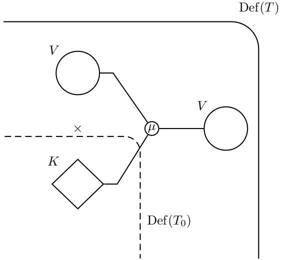


Every $\mathfrak{M} \models T$ induces a $\mathfrak{M}_0 \models T_0$.

Categorical logic

Given a first order theory T

- The category of T-definables (including imaginary sorts) $\mathcal{T} = \mathrm{Def}(T^{eq})$ is a *boolean pre-topos*.
- Models are *logical functors* $\mathfrak{M} : \mathcal{T} \to \mathcal{S}\mathrm{ets}$.
- Interpretations are logical functors $\iota : \mathcal{T}_0 \to \mathcal{T}$.
- ι is an *immersion* if $\iota_X : \mathrm{Sub}_{\mathcal{T}_0}(X) \to \mathrm{Sub}_{\mathcal{T}}(\iota X)$ is an isomorphism for every X.

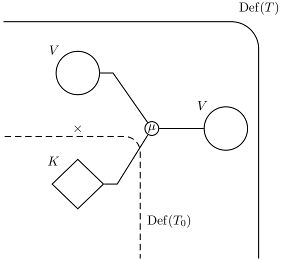


Every $\mathfrak{M} \models T$ induces a $\mathfrak{M}_0 \models T_0$.

Categorical logic

Given a first order theory T

- The category of T-definables (including imaginary sorts) $\mathcal{T} = \mathrm{Def}(T^{eq})$ is a *boolean pre-topos*.
- Models are *logical functors* $\mathfrak{M} : \mathcal{T} \to \mathcal{S}\mathrm{ets}$.
- Interpretations are logical functors $\iota : \mathcal{T}_0 \to \mathcal{T}$.
- ι is an *immersion* if $\iota_X : \mathrm{Sub}_{\mathcal{T}_0}(X) \to \mathrm{Sub}_{\mathcal{T}}(\iota X)$ is an isomorphism for every X.



Every $\mathfrak{M} \models T$ induces a $\mathfrak{M}_0 \models T_0$. In fact, $\iota^* : \mathrm{Mod}(T) \to \mathrm{Mod}(T_0)$ is an equivalence, if and only if, ι is an equivalence.

Definable closure and internal covers

- Given $A \subseteq \mathfrak{M} \models T$, the functor $\mathfrak{A} = \mathrm{dcl}(A)$ preserves limits and co-limits, but not necessarily images (i.e. $\exists$ quantifier).

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\jmath^{\mathfrak{A}}} & \mathcal{T}^{\mathfrak{A}} = \mathrm{Def}(T, A) \\ & {\scriptstyle \mathfrak{A}=\mathrm{dcl}(A)} \searrow & \Big\downarrow {\scriptstyle \Gamma^{\mathfrak{A}}=\mathrm{Hom}(\mathbf{1},?)} \\ & & \mathcal{S}\mathrm{ets} \end{array}$$

- $\iota : \mathcal{T}_0 \to \mathcal{T}$ is a *stable immersion* if $\iota^{\mathfrak{A}}$ is an immersion for every $\mathfrak{A}$.
- Y is T_0-internal over A if for every $\mathfrak{M} \models T^{\mathfrak{A}}$,

$$\mathfrak{M}(Y) = \mathrm{dcl}(\mathfrak{M}_0 \cup A)$$

Definition

ι is an *internal cover* if it's stable and every T-definable is T_0-internal.

Definable closure and internal covers

- Given $A \subseteq \mathfrak{M} \models T$, the functor $\mathfrak{A} = \mathrm{dcl}(A)$ preserves limits and co-limits, but not necessarily images (i.e. $\exists$ quantifier).
- $\iota : \mathcal{T}_0 \to \mathcal{T}$ is a *stable immersion* if $\iota^{\mathfrak{A}}$ is an immersion for every $\mathfrak{A}$.
- Y is T_0-internal over A if for every $\mathfrak{M} \models T^{\mathfrak{A}}$,

$$\mathfrak{M}(Y) = \mathrm{dcl}(\mathfrak{M}_0 \cup A)$$

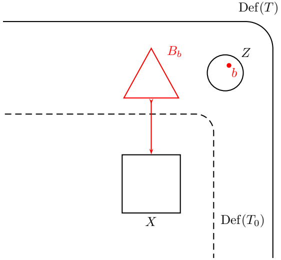
$$\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\jmath^{\mathfrak{A}}} & \mathcal{T}^{\mathfrak{A}} = \mathrm{Def}(T, A) \\
 & \searrow^{\mathfrak{A}=\mathrm{dcl}(A)} & \downarrow \Gamma^{\mathfrak{A}}=\mathrm{Hom}(\mathbf{1},?) \\
 & & \mathcal{S}\mathrm{ets}
\end{array}$$

$$\begin{array}{ccc}
\mathcal{T}_0 & \xrightarrow{\jmath^{\mathfrak{A}_0}} & \mathcal{T}_0^{\mathfrak{A}_0} \\
\downarrow \iota & & \downarrow \iota^{\mathfrak{A}} \\
\mathcal{T} & \xrightarrow{\jmath^{\mathfrak{A}}} & \mathcal{T}^{\mathfrak{A}}
\end{array}$$

Definable closure and internal covers

- Given $A \subseteq \mathfrak{M} \models T$, the functor $\mathfrak{A} = \mathrm{dcl}(A)$ preserves limits and co-limits, but not necessarily images (i.e. $\exists$ quantifier).
- $\iota : \mathcal{T}_0 \to \mathcal{T}$ is a *stable immersion* if $\iota^{\mathfrak{A}}$ is an immersion for every $\mathfrak{A}$.
- Y is T_0-internal over A if for every $\mathfrak{M} \models T^{\mathfrak{A}}$,

$$\mathfrak{M}(Y) = \mathrm{dcl}(\mathfrak{M}_0 \cup A)$$



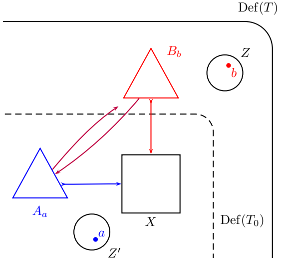
Definition

ι is an *internal cover* if it's stable and every T-definable is T_0-internal.

Definable closure and internal covers

- Given $A \subseteq \mathfrak{M} \models T$, the functor $\mathfrak{A} = \mathrm{dcl}(A)$ preserves limits and co-limits, but not necessarily images (i.e. $\exists$ quantifier).
- $\iota : \mathcal{T}_0 \to \mathcal{T}$ is a *stable immersion* if $\iota^{\mathfrak{A}}$ is an immersion for every $\mathfrak{A}$.
- Y is T_0-internal over A if for every $\mathfrak{M} \models T^{\mathfrak{A}}$,

$$\mathfrak{M}(Y) = \mathrm{dcl}(\mathfrak{M}_0 \cup A)$$



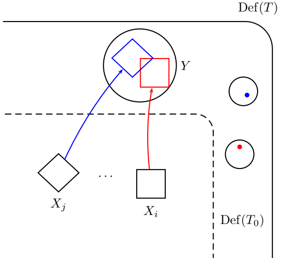
Definition

ι is an *internal cover* if it's stable and every T-definable is T_0-internal.

Definable closure and internal covers

- Given $A \subseteq \mathfrak{M} \models T$, the functor $\mathfrak{A} = \mathrm{dcl}(A)$ preserves limits and co-limits, but not necessarily images (i.e. $\exists$ quantifier).
- $\iota : \mathcal{T}_0 \to \mathcal{T}$ is a *stable immersion* if $\iota^{\mathfrak{A}}$ is an immersion for every $\mathfrak{A}$.
- Y is T_0-internal over A if for every $\mathfrak{M} \models T^{\mathfrak{A}}$,

$$\mathfrak{M}(Y) = \mathrm{dcl}(\mathfrak{M}_0 \cup A)$$

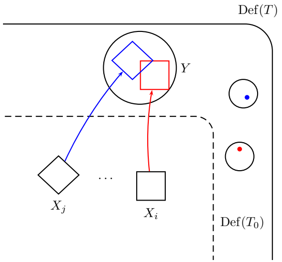


Definition
ι is an *internal cover* if it's stable and every T-definable is T_0-internal.

Definable closure and internal covers

- Given $A \subseteq \mathfrak{M} \models T$, the functor $\mathfrak{A} = \mathrm{dcl}(A)$ preserves limits and co-limits, but not necessarily images (i.e. $\exists$ quantifier).
- $\iota : \mathcal{T}_0 \to \mathcal{T}$ is a *stable immersion* if $\iota^{\mathfrak{A}}$ is an immersion for every $\mathfrak{A}$.
- Y is T_0-internal over A if for every $\mathfrak{M} \models T^{\mathfrak{A}}$,

$$\mathfrak{M}(Y) = \mathrm{dcl}(\mathfrak{M}_0 \cup A)$$



Definition

ι is an *internal cover* if it's stable and every T-definable is T_0-internal.

Monadicity and definability of the binding group

Lemma (Kamensky)

$\iota : \mathcal{T}_0 \to \mathcal{T}$ is a stable immersion iff $\operatorname{Ind}\iota : \operatorname{Ind}\mathcal{T}_0 \to \operatorname{Ind}\mathcal{T}$ is a cartesian closed functor.

Proof.

Use that $\mathbf{2} = \mathbf{1} + \mathbf{1}$ is the subobject classifier. $\square$

Lemma

If $\mathfrak{A} : \mathcal{T}_0 \to \mathcal{S}\text{ets}$ contains a basis for the internal cover ι (i.e. every Y is T_0-internal over $\mathfrak{A}$), then $\iota^{\mathfrak{A}}$ is an equivalence.

Lemma

If T is a complete theory, for every $\mathfrak{A}$ the functor $\operatorname{Pro}\jmath^{\mathfrak{A}} : \operatorname{Pro}\mathcal{T} \to \operatorname{Pro}\mathcal{T}^{\mathfrak{A}}$ is monadic.

Monadicity and definability of the binding group

Lemma (Kamensky)

$\iota : \mathcal{T}_0 \to \mathcal{T}$ is a stable immersion iff $\operatorname{Ind} \iota : \operatorname{Ind} \mathcal{T}_0 \to \operatorname{Ind} \mathcal{T}$ is a cartesian closed functor.

Lemma

If $\mathfrak{A} : \mathcal{T}_0 \to \mathcal{S}\text{ets}$ contains a basis for the internal cover ι (i.e. every Y is T_0-internal over $\mathfrak{A}$), then $\iota^{\mathfrak{A}}$ is an equivalence.

Proof.

Use compactness (and co-products) to get a regular epi $f_a : \iota X \to Y$, afterwards use stability (and effectiveness) to define $\phi_a : \iota(X/E_b) \to Y$. □

Lemma

If T is a complete theory, for every $\mathfrak{A}$ the functor $\operatorname{Pro} \iota^{\mathfrak{A}} : \operatorname{Pro} \mathcal{T} \to \operatorname{Pro} \mathcal{T}^{\mathfrak{A}}$ is monadic.

Monadicity and definability of the binding group

Lemma (Kamensky)

$\iota : \mathcal{T}_0 \to \mathcal{T}$ is a stable immersion iff $\operatorname{Ind}\iota : \operatorname{Ind}\mathcal{T}_0 \to \operatorname{Ind}\mathcal{T}$ is a cartesian closed functor.

Lemma

If $\mathfrak{A} : \mathcal{T}_0 \to \mathcal{S}ets$ contains a basis for the internal cover ι (i.e. every Y is T_0-internal over $\mathfrak{A}$), then $\iota^{\mathfrak{A}}$ is an equivalence.

Lemma

If T is a complete theory, for every $\mathfrak{A}$ the functor $\operatorname{Pro}\jmath^{\mathfrak{A}} : \operatorname{Pro}\mathcal{T} \to \operatorname{Pro}\mathcal{T}^{\mathfrak{A}}$ is monadic.

Proof.

Use Duskin variant of Beck's monadicity □

Monadicity and definability of the binding group

Lemma (Kamensky)

$\iota : \mathcal{T}_0 \to \mathcal{T}$ is a stable immersion iff $\operatorname{Ind} \iota : \operatorname{Ind} \mathcal{T}_0 \to \operatorname{Ind} \mathcal{T}$ is a cartesian closed functor.

Lemma

If $\mathfrak{A} : \mathcal{T}_0 \to \mathcal{S}\mathrm{ets}$ contains a basis for the internal cover ι (i.e. every Y is T_0-internal over $\mathfrak{A}$), then $\iota^{\mathfrak{A}}$ is an equivalence.

Lemma

If T is a complete theory, for every $\mathfrak{A}$ the functor $\operatorname{Pro} \jmath^{\mathfrak{A}} : \operatorname{Pro} \mathcal{T} \to \operatorname{Pro} \mathcal{T}^{\mathfrak{A}}$ is **Hopf** *monadic.*

Proof.

See [BLV11] with a caveat. □

Galois theory of *neutral* internal covers

Theorem

Let $\iota : \mathcal{T}_0 \to \mathcal{T}$ *be an internal cover neutralized by* $\mathfrak{A}$*. (i.e.* $\mathfrak{A}$ *contains an internality basis and* $\mathfrak{A}_0 = \Gamma_0 = \mathrm{dcl}_{\mathcal{T}_0}(0)$ *) There is a pro-group* $G = \mathrm{Pro}\, G_k$ *in* $\mathrm{Pro}\, \mathcal{T}_0$ *and an equivalence between* $\mathcal{T}$ *and* $\mathcal{T}_0^G$*.*

Galois theory of *neutral* internal covers

Theorem

Let $\iota : \mathcal{T}_0 \to \mathcal{T}$ be an internal cover neutralized by $\mathfrak{A}$. There is a pro-group $G = \operatorname{Pro} G_k$ in $\operatorname{Pro} \mathcal{T}_0$ and an equivalence between $\mathcal{T}$ and $\mathcal{T}_0^G$.

Proof.

By [BLV11], augmented Hopf monads are $\otimes$-representable by Hopf monoids. □

Galois theory of *neutral* internal covers

Theorem

Let $\iota : \mathcal{T}_0 \to \mathcal{T}$ *be an internal cover neutralized by* $\mathfrak{A}$. *There is a pro-group* $G = \operatorname{Pro} G_k$ *in* $\operatorname{Pro} \mathcal{T}_0$ *and an equivalence between* $\mathcal{T}$ *and* $\mathcal{T}_0^G$.

Corollary

For every $\mathfrak{M} \models T^{\mathfrak{A}}$, $\operatorname{Aut}(\mathfrak{M}/\mathfrak{M}_0)) \simeq_{\mathfrak{A}} \varprojlim \mathfrak{M}_0(G_k)$.

Galois theory of *neutral* internal covers

Theorem

Let $\iota : \mathcal{T}_0 \to \mathcal{T}$ *be an internal cover neutralized by* $\mathfrak{A}$. *There is a pro-group* $G = \operatorname{Pro} G_k$ *in* $\operatorname{Pro} \mathcal{T}_0$ *and an equivalence between* $\mathcal{T}$ *and* $\mathcal{T}_0^G$.

Corollary

For every $\mathfrak{M} \models T^{\mathfrak{A}}$, *there are biyective correspondences between:*

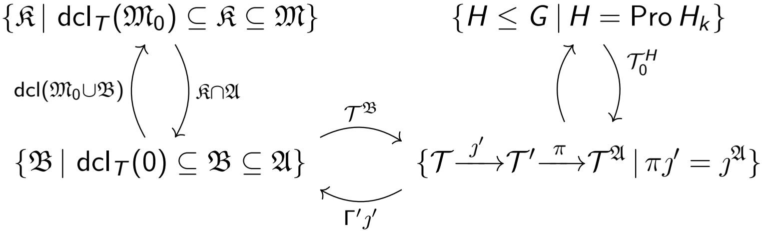
$$\mathrm{Aut}(\mathfrak{M}/\mathfrak{K}) \simeq \varprojlim \mathfrak{M}_0(H_l)$$

$$\{\mathfrak{K} : \mathcal{T} \to \mathcal{S}\mathrm{ets} \mid \mathrm{dcl}_T(\mathfrak{M}_0) \subseteq \mathfrak{K} \subseteq \mathfrak{M}\} \qquad \{H \leq G \mid H = \operatorname{Pro} H_k\}$$

$$\mathfrak{M}^{\mathfrak{M}_0(H_k)}$$

Galois theory of *neutral* internal covers

Proof.



where the $\jmath'$ are stable embeddings, therefore $\jmath' \iota : \mathcal{T}_0 \to \mathcal{T}'$ is an internal cover *neutralized* by $\mathfrak{A}$. □

References

- Alain Bruguières, Steve Lack, and Alexis Virelizier, *Hopf monads on monoidal categories*, Advances in Mathematics **227** (2011), no. 2, 745–800.
- Alexander Grothendieck, *Revêtements étales et groupe fondamental (SGA 1)*, Lecture Notes in Mathematics, vol. 224, Springer, 1971.
- Moshe Kamensky, *A categorical approach to internality*, Models, logics, and higher-dimensional categories (Bradd Hart et al., eds.), CRM proc. & lecture notes, vol. 53, American Math. Soc., 2011, pp. 139–156.
- ______, *Model theory and the tannakian formalism*, Transactions of the American Math. Soc. **367** (2015), no. 2, 1095–1120.