Diaconescu's Theorem for Stacks

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with G^* left exact (finite limit preserving).

A functor $E \colon \mathscr{C} \to \mathbf{Set}$ is flat if the extension

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\quad E \quad} & \mathbf{Set}. \\ {\scriptstyle \mathbf{y}} \downarrow & \circlearrowright & \nearrow {\scriptstyle E \otimes_{\mathscr{C}} -} \\ [\mathscr{C}^{op}, \mathbf{Set}] & & \end{array}$$

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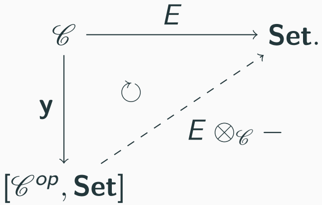

is left exact. The tensor is a left adjiont

$$E \otimes_{\mathscr{C}} - \dashv \mathbf{Set}(E, -)$$

hence each such E yields a geometric morphism.

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The general version is found in

R. Diaconescu. "Change of Base for Toposes with Generators." *Jour. Pure Appl. Alg.* 6 (1975), pp. 191-218.

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The previous equivalence (roughly speaking) restricts to one

$$\mathbf{ConFlat}(\mathscr{C}, J) \simeq \mathbf{Geom}(\mathbf{Set}, \mathbf{Sh}(\mathscr{C}, J)).$$

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Question/problem: What sort of correspondence exists between continuous flat pseudo-functors $E\colon \mathscr{C} \to \mathfrak{CAT}$ and points of stacks?

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One reference: §§4-5 of

R. Street, "Two-Dimensional Sheaf Theory." *Jour. Pure Appl. Alg.* 23 (1982), pp. 251-270.

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Take $E \otimes_{\mathscr{C}} F$ to denote the category of fractions

$$E \otimes_{\mathscr{C}} F := \Delta(E, F)[\Sigma^{-1}]$$

where Σ is the set of “cartesian” morphisms.

There results an extension

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For a study of flat pseudo-functors see

M.E. Descotte, E.J. Dubuc, M. Szyld. "Sigma limits in 2-categories and flat pseudofunctors." arXiv:1610.09429v3.

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Crucial part: there is an equivalence

$$G^*\mathbf{y} \otimes_{\mathscr{C}} F \simeq G^* F.$$

pseudo-natural in F.

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The work consists in showing that

$$\mathfrak{Fact} \simeq \mathfrak{Geom}(\mathfrak{CAT}, \mathfrak{St}(\mathscr{C}, J)).$$

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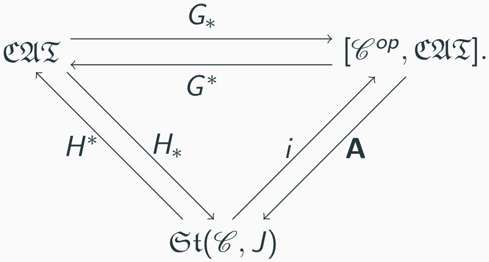

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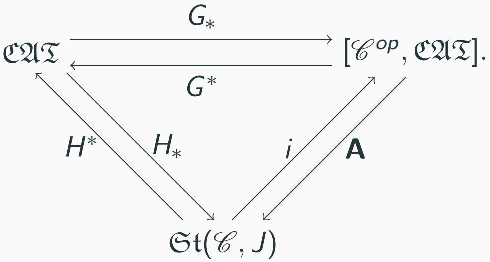

Thus, have that

$$H^* \simeq G^* i \qquad\qquad H_* \simeq \mathbf{A} G_*.$$

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Theorem

There is a biequivalence of 2-categories

$$\mathfrak{ConFlat}(\mathscr{C}, J) \simeq \mathfrak{Geom}(\mathfrak{CAT}, \mathfrak{St}(\mathscr{C}, J)).$$