The homotopy relation in a category with weak equivalences

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University of Buenos Aires - CONICET, Argentina

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Model (bi)categories:

a structure $(\mathcal{C}, \mathcal{F}, co\mathcal{F}, \mathcal{W})$, with $\mathcal{C}$ a (bi)category, and

$\mathcal{F}$	$co\mathcal{F}$	$\mathcal{W}$	families of arrows of $\mathcal{C}$
$\cdot \twoheadrightarrow \cdot$	$\cdot \hookrightarrow \cdot$	$\cdot \xrightarrow{\sim} \cdot$	satisfying some axioms.

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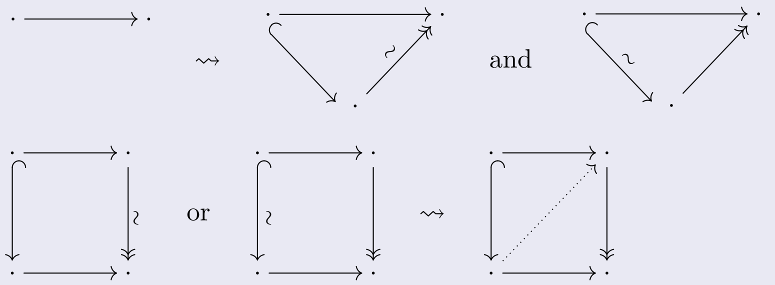

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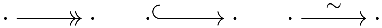

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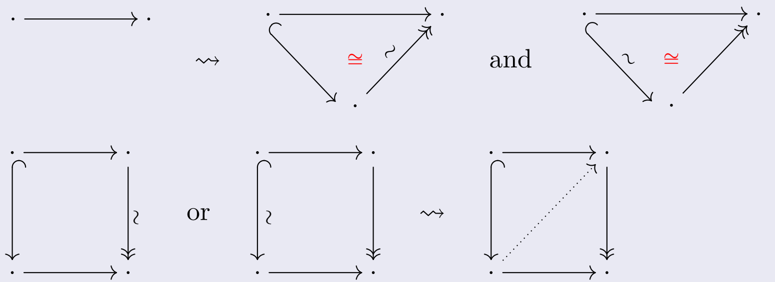

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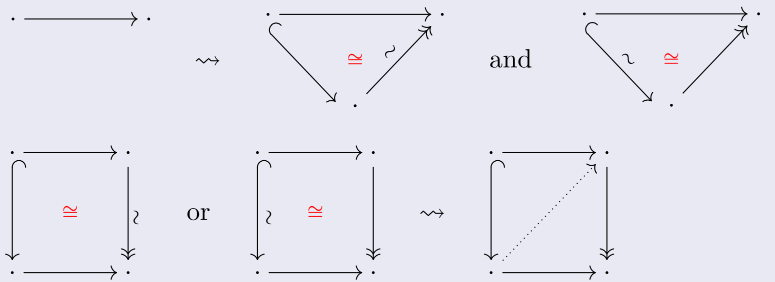

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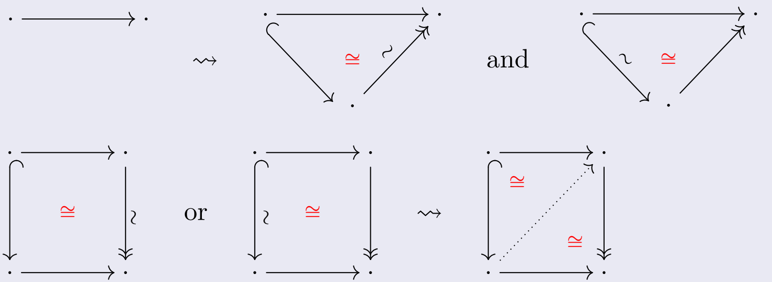

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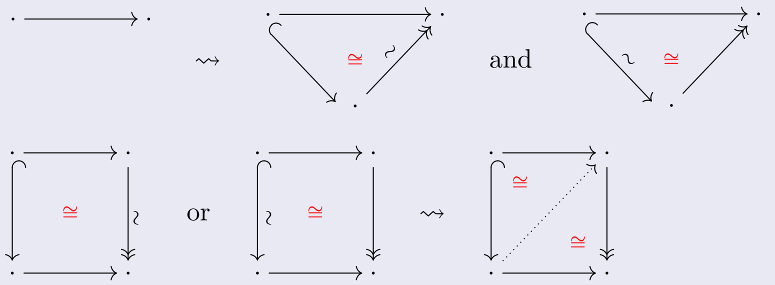

$\mathrm{Ho}(\mathcal{C}) = \mathcal{C}[\mathcal{W}^{-1}]$ admits a construction "quotienting by homotopy".

Our original problem: homotopy in a model bicategory

We[1] seek a construction of the homotopy bicategory $\mathcal{H}o(\mathcal{C})$:

- Objects and arrows are those of $\mathcal{C}_{fc}$ ($0 \hookrightarrow X \twoheadrightarrow 1$).
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Simultaneous requirements

- Vertical composition
- Horizontal composition

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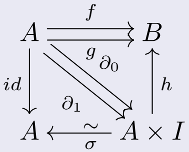
Considering Quillen's notion $\rightsquigarrow$ an obstacle

$f \overset{\ell}{\sim} g$ if and only if there is a diagram in which σ is a weak equivalence (and $A \amalg A \xrightarrow{\partial_0 + \partial_1} A \times I$ is a cofibration)



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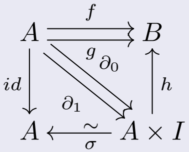

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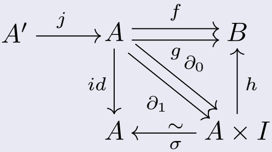

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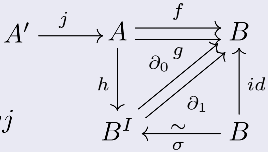

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Homotopy in a category with weak equivalences

Quote from [DHKS] book

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2. Explicit construction of $\sim_{\mathcal{W}}$, similar to $\overset{\ell}{\sim}$
3. For $\mathcal{C}$ model: $(\mathcal{C}_{fc}, \mathcal{W})$ satisfies this condition, and $\sim_{\mathcal{W}} = \overset{\ell}{\sim}$

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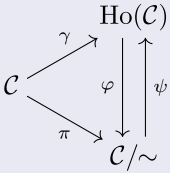

If $\mathcal{C}/\!\sim\, = \mathrm{Ho}(\mathcal{C})$, then $\sim$ has to be $\sim_{\mathcal{W}}$:

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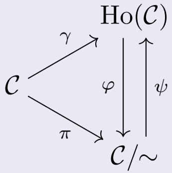

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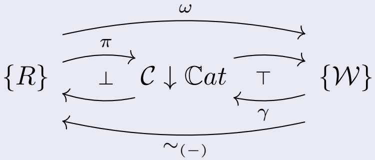

$\mathcal{C}/R = \mathrm{Ho}(\mathcal{C})$ if and only if

(1) $\mathcal{W} \subseteq \omega R$ and $R \subseteq \sim_{\mathcal{W}}$

Fix $\mathcal{W}$. Then $\mathcal{C}/\!\sim_{\mathcal{W}} = \mathrm{Ho}(\mathcal{C})$ if and only if (2) $\mathcal{W} \subseteq \omega \sim_{\mathcal{W}}$.

The Whitehead condition

ωR is the family of R-equivalences (arrows that admit an R-inverse).

(2) $\mathcal{W} \subseteq \omega \sim_{\mathcal{W}}$: any w.e. is a homotopical equivalence.

We say that such a $(\mathcal{C}, \mathcal{W})$ is *Whitehead*.

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The Whitehead condition in model categories

Prop: Split-generated ⇒ Whitehead.

Proof: Because split w.e. are homotopical equivalences:

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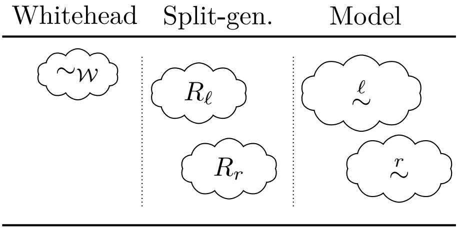
  (any w.e. is a section followed by a retraction, both w.e.)
- It follows $\mathcal{C}_{fc}/\sim_{\mathcal{W}} = \text{Ho}(\mathcal{C}_{fc})$.
- Recall that $\sim_{\mathcal{W}}$ is the only possible congruence such that this equality holds.

The congruence $\sim_{\mathcal{W}}$ can be constructed from different R satisfying (1)



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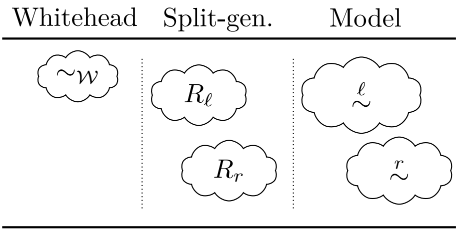

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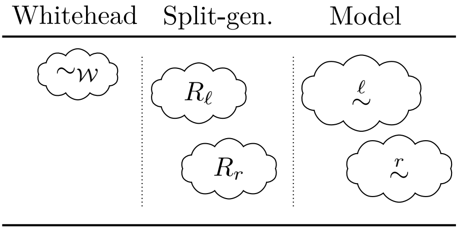

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- $R_\ell \subseteq \sim_{\mathcal{W}}$ ✓

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Whitehead	Split-gen.	Model	
$\sim_{\mathcal{W}}$	R_ℓ	$\overset{\ell}{\sim}$	$R \subseteq \sim_{\mathcal{W}}$
	R_r	$\overset{r}{\sim}$	
			$\mathcal{W} \subseteq \omega R$, "$R$ inverts $\mathcal{W}$"

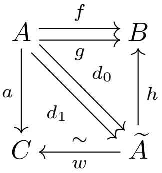
$f R_\ell\, g$ if and only if $\cdot \underset{g}{\overset{f}{\rightrightarrows}} \cdot \xrightarrow[w]{\sim} \cdot$ $(wf = wg,\ w \text{ w.e.})$

- $R_\ell \subseteq \sim_{\mathcal{W}}$ ✓
- R_ℓ inverts split w.e. $\begin{cases} rs = id \ \Rightarrow \ rsR_\ell\, id \\ rsr = r \ \Rightarrow \ srR_\ell\, id \end{cases}$

A construction of $\sim_{\mathcal{W}}$ from R_ℓ

First we close R_ℓ by composition, then by transitivity.

$f \, R_\ell^c \, g$:



R_ℓ^c is a *relaxed* version of $\overset{\ell}{\sim}$ in which $A \xrightarrow{id} A$ can be any arrow a.

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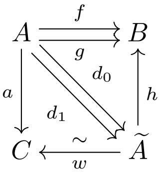
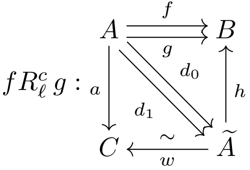


R^c_ℓ is a *relaxed* version of $\overset{\ell}{\sim}$ in which $A \xrightarrow{id} A$ can be any arrow a.

$f \sim_{\mathcal{W}} g$ if and only if $f R^c_\ell f_1 R^c_\ell \dots f_n R^c_\ell g$.

A construction of $\sim_{\mathcal{W}}$ from R_ℓ

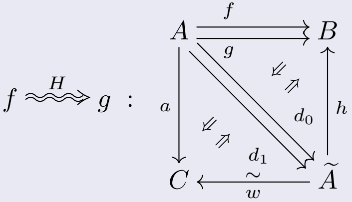
First we close R_ℓ by composition, then by transitivity.



R^c_ℓ is a *relaxed* version of $\overset{\ell}{\sim}$ in which $A \xrightarrow{id} A$ can be any arrow a.

$f \sim_{\mathcal{W}} g$ if and only if $f R^c_\ell\, f_1 R^c_\ell \ldots f_n R^c_\ell\, g$.

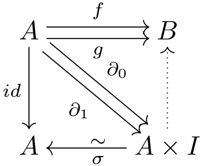
In dimension 2



"homotopy respect to the w.e." behaves better for forming the 2-cells of $\mathcal{H}o(\mathcal{C})$.

The case of model categories

Prop: If $f\, R_\ell^c\, g$ then for any cylinder object,



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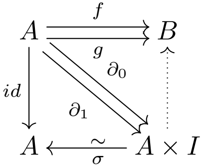
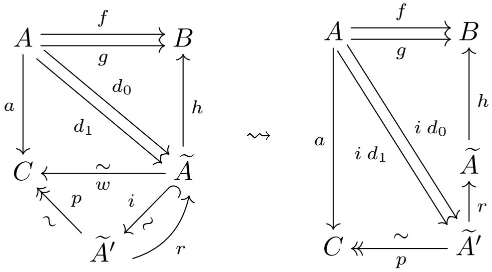


Proof: in 2 steps. Step 1: In $f\,R^c_\ell\,g$ we may assume w a fibration



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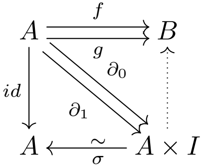

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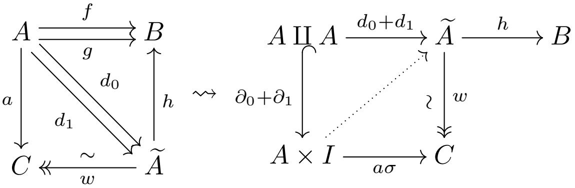

$\square$

The case of model categories

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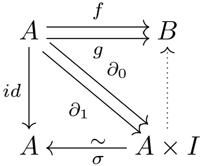

Consequences:

1. $R^c_\ell = \overset{\ell}{\sim} = \sim_{\mathcal{W}}$, in particular we recover $\mathcal{C}_{fc}/\overset{\ell}{\sim} = \mathrm{Ho}(\mathcal{C}_{fc})$.

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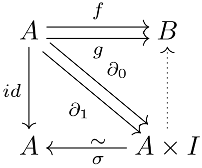
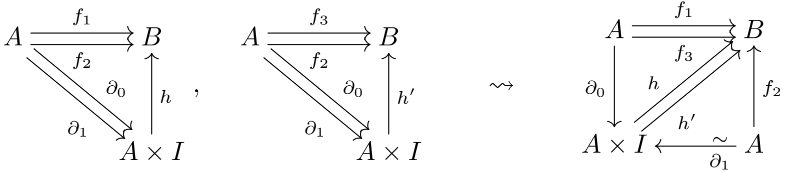


Consequences:

1. $R^c_\ell = \overset{\ell}{\sim} = \sim_{\mathcal{W}}$, in particular we recover $\mathcal{C}_{fc}/\overset{\ell}{\sim} = \mathrm{Ho}(\mathcal{C}_{fc})$.
2. New proofs of $\overset{\ell}{\sim} = \overset{r}{\sim}$ and of transitivity, both follow from:

$$f_1 \overset{\ell}{\sim} f_2, \qquad f_2 \overset{\ell}{\sim} f_3 \qquad \Rightarrow \qquad f_1 \overset{r}{\sim} f_3$$



Further Results

- Fibrant-cofibrant replacement in this context.
- Analysis of the saturated condition in this case.
  Corollary: any model category is saturated.

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References

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Thank you!