On the Operational Meaning of the Bar Construction

...with an application to Probability

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Leipzig, Germany

Category Theory 2018

Monads and formal expressions

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$$f : X \to Y \qquad \longmapsto \qquad Tf\ :\ x + x' \mapsto f(x) + f(x')$$

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The bar construction

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$$TA \xrightarrow{e} A$$

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$$TTA \underset{T\eta}{\overset{Te}{\underset{\mu}{\overset{\curvearrowright}{\longrightarrow}}}} TA \xrightarrow{e} A$$

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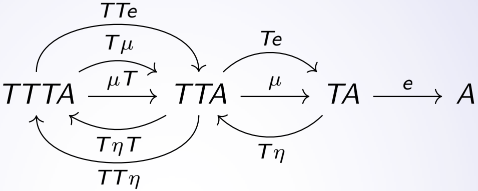

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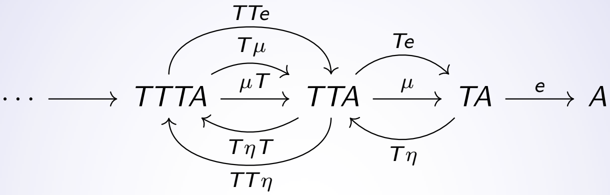

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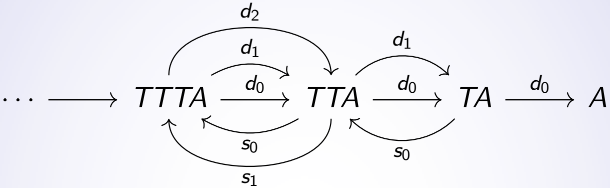

Simplicial object:

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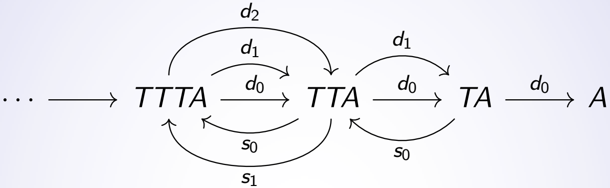

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- A monad defines a comonad on the category of algebras C^T

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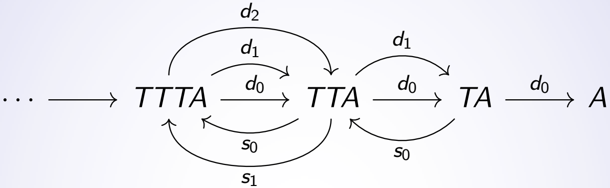

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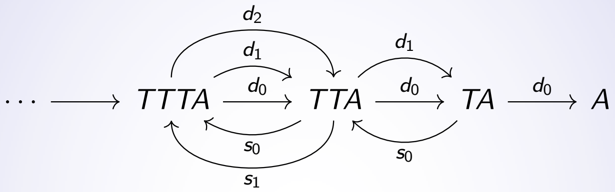

Simplicial object:

- A monad defines a comonad on the category of algebras C^T
- A comonad is a comonoid in $[\mathsf{C}^T, \mathsf{C}^T]$
- A comonoid is a (monoidal) functor ${\Delta_\mathsf{a}}^{\mathrm{op}} \to [\mathsf{C}^T, \mathsf{C}^T]$.

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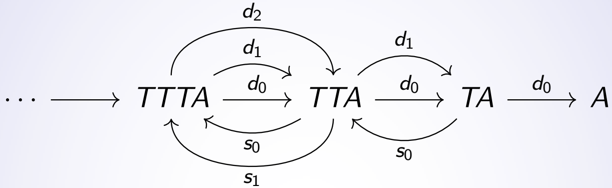

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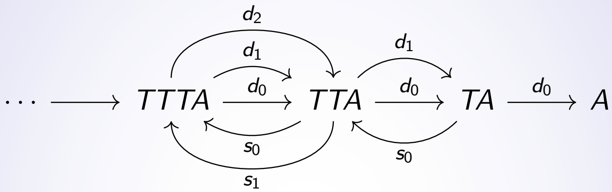

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- How can we interpret all these extra objects and arrows?

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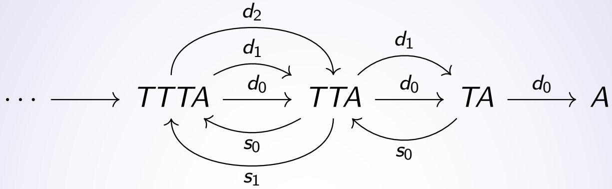

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- Can we interpret *the whole simplicial object* operationally?

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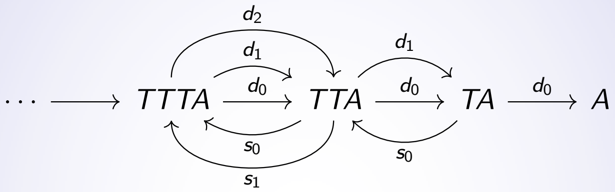

Questions:

- How can we interpret all these extra objects and arrows?
- Can we interpret *the whole simplicial object* operationally?
- Can this be applied to other areas of math?

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A formal expression of elements of an algebra can also be *partially evaluated*, instead of *totally*.

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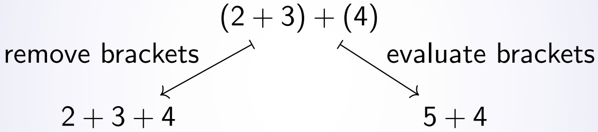
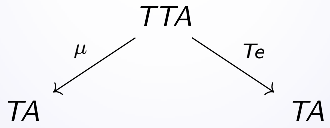
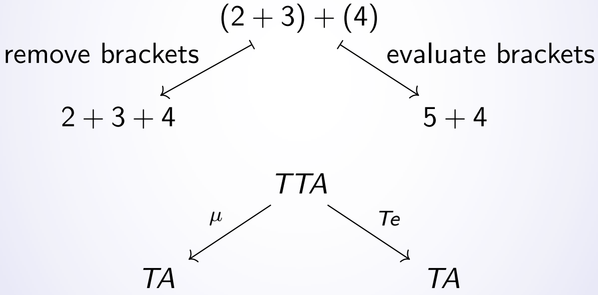




Partial evaluations

Definition:
Let $p, q \in TA$. A *partial evaluation* from p to q is an element $m \in TTA$ such that $\mu(m) = p$ and $(Te)(m) = q$.



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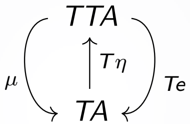

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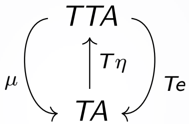

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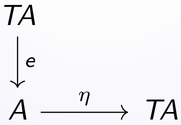

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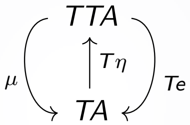

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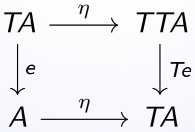

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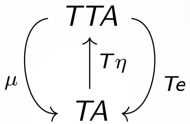

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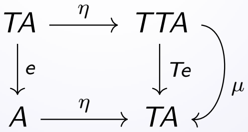

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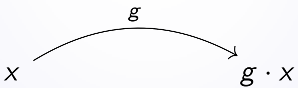

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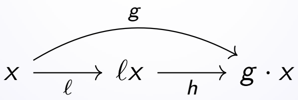

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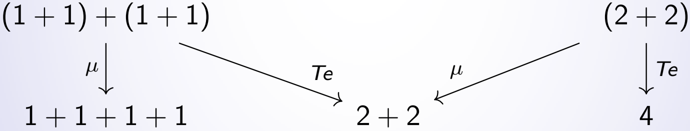

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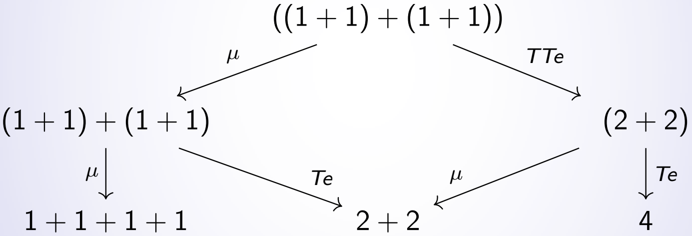

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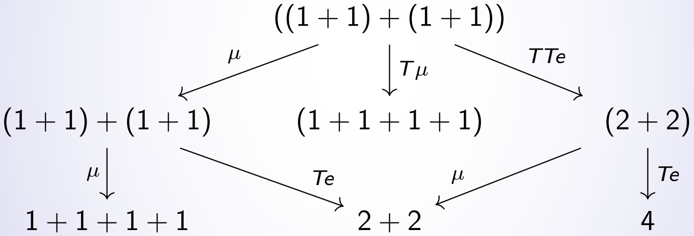

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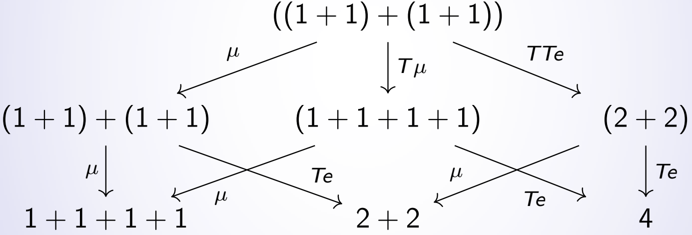

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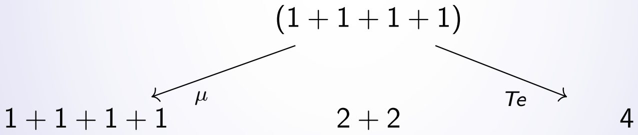

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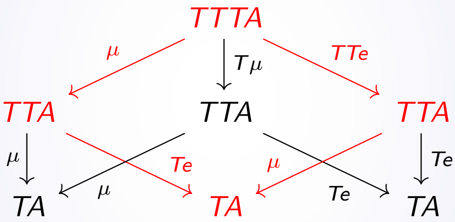

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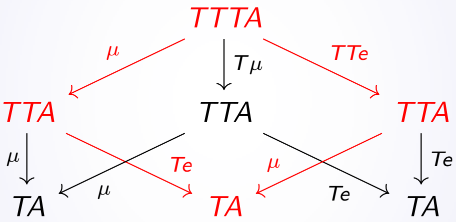

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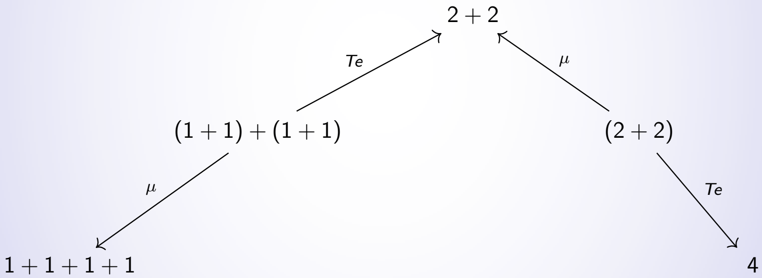

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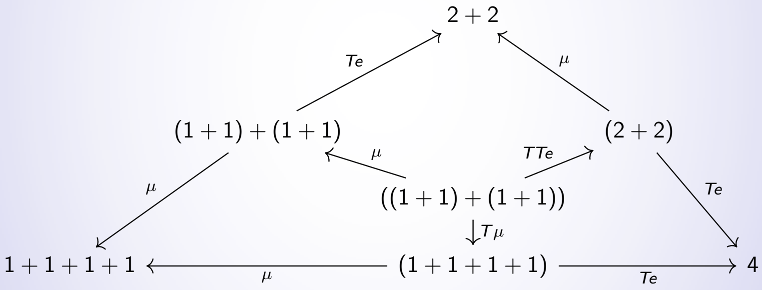

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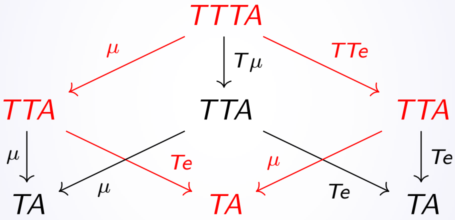

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$$(4(-1)) + (4(+1)) + 2(2(-2) + 2(+2))$$

$$(4(-1) + 4(+1)) + (3(-2) + (+2)) + ((-2) + 3(+2))$$

These give unequal parallel 1-cells between:

$$4(-1) + 4(+1) + 4(-2) + 4(+2) \quad \text{and} \quad (+4) + (-4).$$

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3. Is there a link with generalized multicategories?

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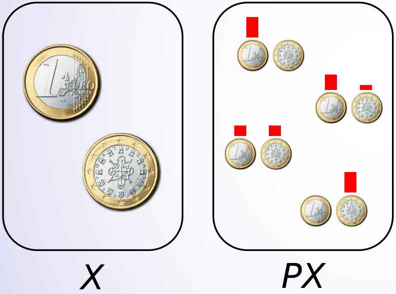

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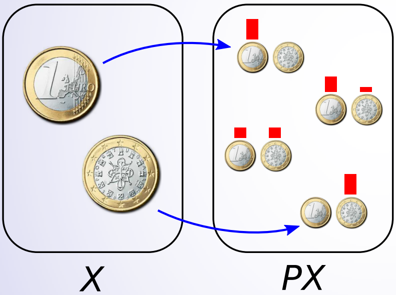

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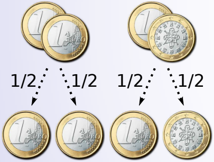

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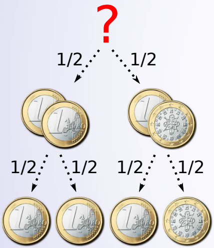

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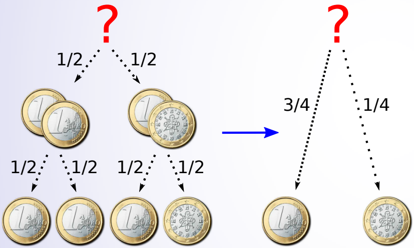

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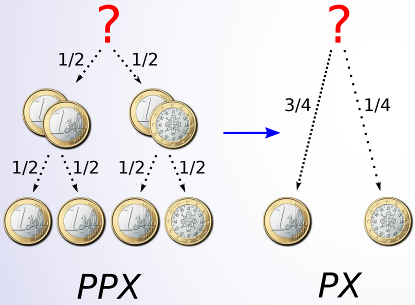

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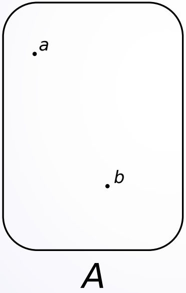

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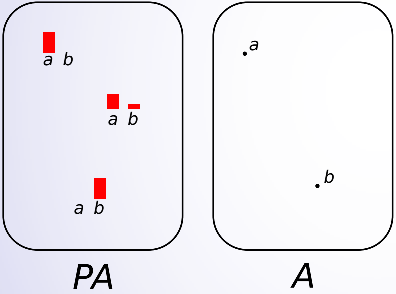

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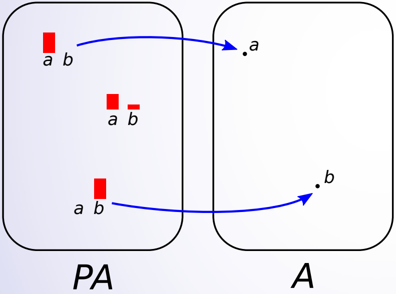

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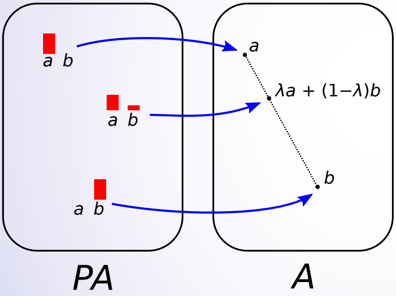

- Algebras $e : PA \to A$ are "convex spaces"
- Formal averages are mapped to actual averages

The Kantorovich monad

Kantorovich monad [van Breugel, 2005, Fritz and Perrone, 2017]:

- Given a complete metric space X, PX is the set of Radon probability measures of finite first moment, equipped with the *Wasserstein distance*, or *Kantorovich-Rubinstein distance*, or *earth mover's distance*:

$$d_{PX}(p,q) = \sup_{f:X\to\mathbb{R}} \left| \int_X f(x)\, d(p-q)(x) \right|$$

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- Algebras of P are closed convex subsets of Banach spaces.

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Partial evaluations for P are “partial expectations”.

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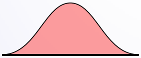

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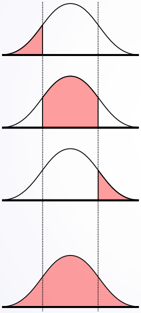

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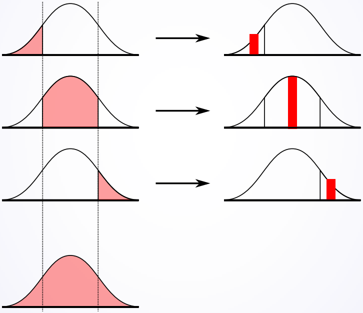

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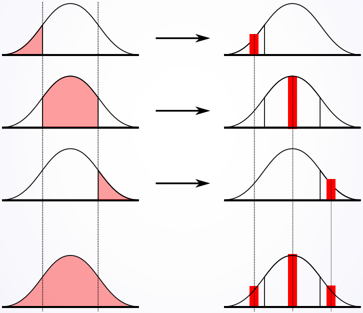

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2. *Partial expectations can always be composed (not uniquely)*;
3. The relation "Admitting a partial evaluation" is a closed partial order, which we call *partial evaluation order*. This is a (0,1)-truncation of the bar construction for P.

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Theorem, extending [Winkler, 1985, Theorem 1.3.6]

Let A be a P-algebra and $p, q \in PA$. The following conditions are equivalent:

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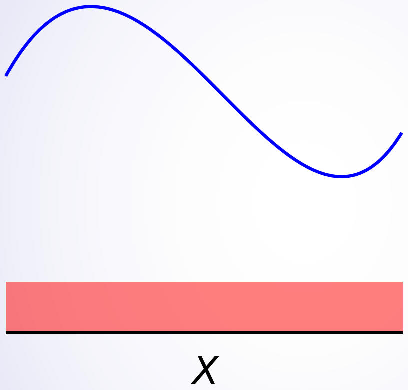

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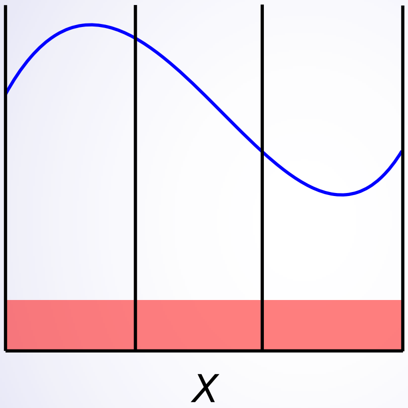

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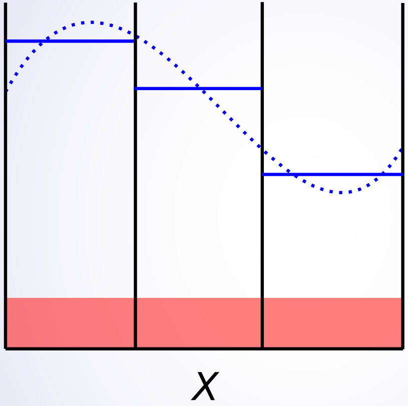

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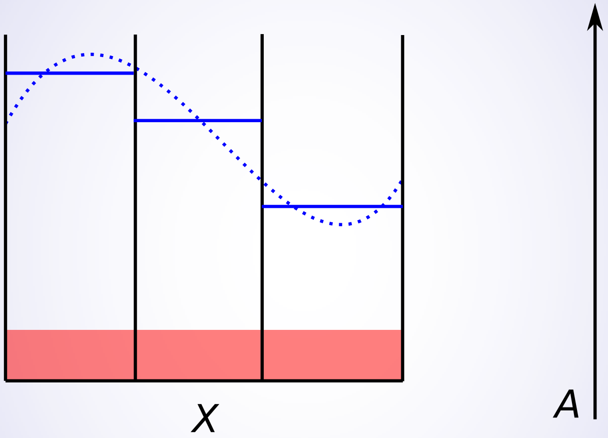

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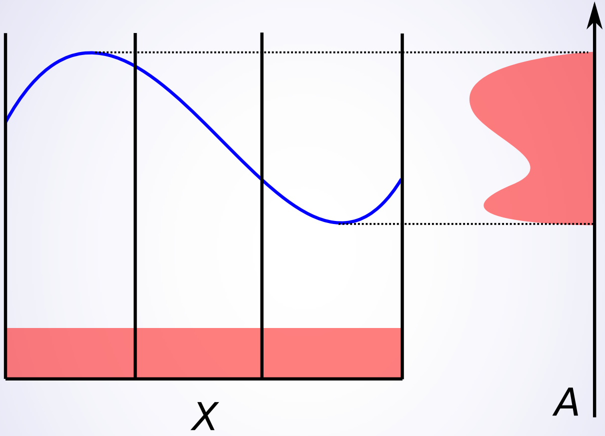

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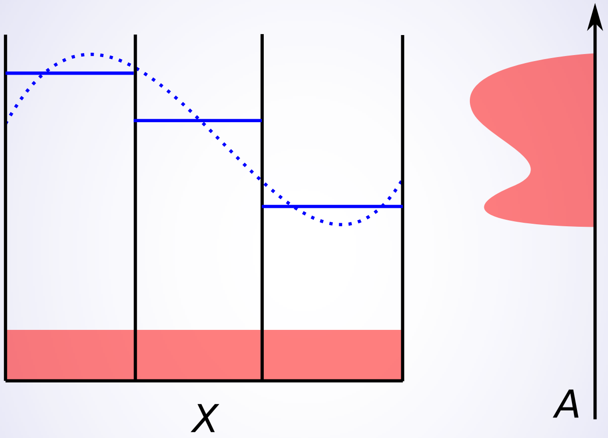

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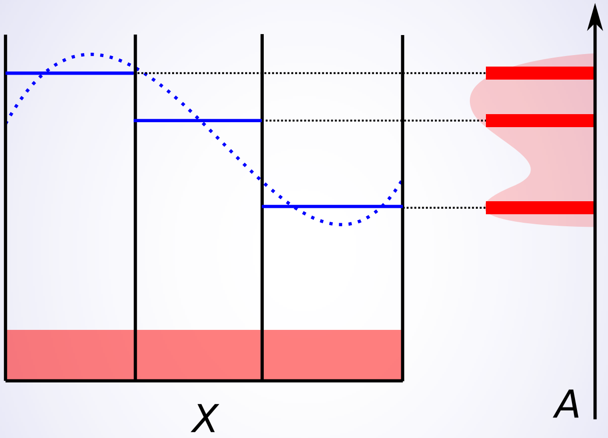

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Corollary

A chain of composable partial evaluations in PA is (basically) the same as a *martingale* on A, in reverse time.

Ordered Kantorovich monad

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Definition

An *L-ordered metric space* is a metric space X equipped with a partial order such that for all x, y, the following are equivalent:

- $x \leq y$
- For all short, monotone $f : X \to \mathbb{R}$, $f(x) \leq f(y)$.

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Definition (stochastic order)

Let $p, q \in PX$. We say that $p \leq q$ if equivalently:

- There exists a coupling of p and q entirely supported on $\{x \leq y\} \in X \times X$;
- For all short, monotone $f : X \to \mathbb{R}$, $\int f\, dp \leq \int f\, dq$.

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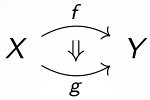
Theorem

- If X is L-ordered, PX with the stochastic order is L-ordered;
- P lifts to a monad on the category L-COMet of L-ordered spaces;
- The algebras of P are exactly closed convex subsets of ordered Banach spaces (i.e. equipped with a closed positive cone).

Ordered Kantorovich monad

Pointwise order: $f \leq g : X \to Y$ iff for every $x \in X$, $f(x) \leq g(x)$.



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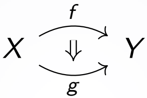

Proposition:

Let $f \leq g : X \to Y$. Then $Pf \leq Pg : PX \to PY$.

Corollary:

L-COMet is a strict 2-category, and P a strict 2-monad.

Lax codescent objects

Proposition:

Let A be a (unordered) P-algebra in L-COMet. The partial evaluation order on PA is the coinserter in L-COMet^P of the diagram:

$$PPA \underset{Pe}{\overset{E}{\rightrightarrows}} PA.$$

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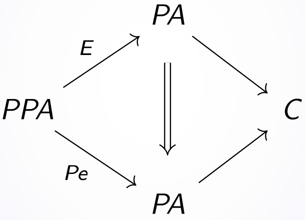

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Corollary:

Let A be ordered. The lax codescent object obtained as above gives again PA, with as order the composition of:

- The partial evaluation order, and
- The stochastic order on PA induced by the order on A.

Let's call this order $(PA, \preceq_\ell)$, *lax codescent order*.

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Let A be a (unordered) P-algebra in L-COMet. The partial evaluation order on PA is the lax codescent object of the algebra A.

Explicitly:

Given $p, q \in PA$, $p \preceq_\ell q$ if and only if there exists $p' \in PA$ such that which can be obtained by partially averaging p, and such that $p' \leq q$ in the stochastic order.

$$p \xmapsto{\text{p. eval.}} p' \xrightarrow{\leq} q$$

Lax codescent objects

The adjunction associated to P is natural isomorphism of partial orders:

$$\text{L-COMet}(X, B) \xrightarrow{\cong} \text{L-COMet}^P(PX, B)$$

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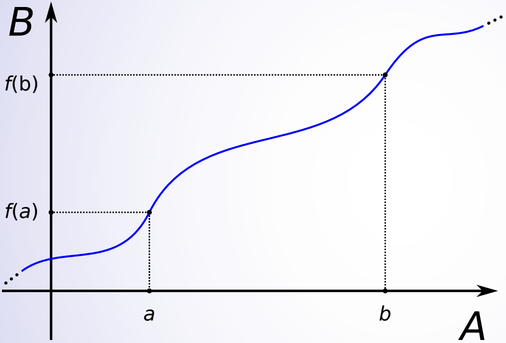
Theorem (Corollary of [Lack, 2002]):

Let A and B and be a P-algebras. The adjunction above specializes to:

$$\text{L-COMet}^P_{lax}(A, B) \cong \text{L-COMet}^P\left((PA, \preceq_\ell), B\right).$$

Lax codescent objects



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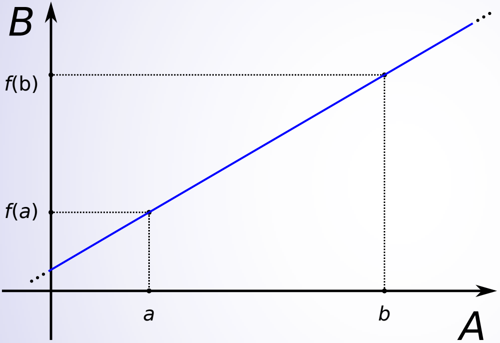

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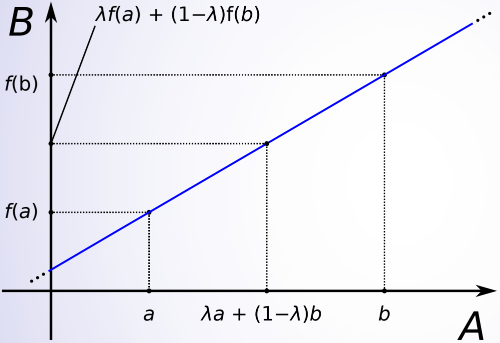

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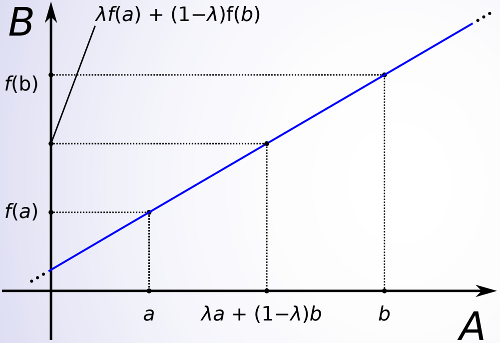

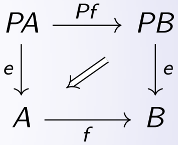

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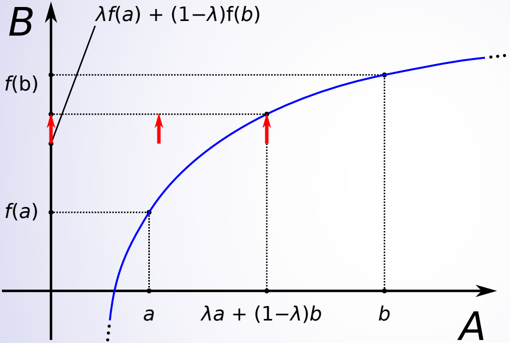

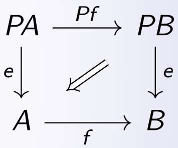

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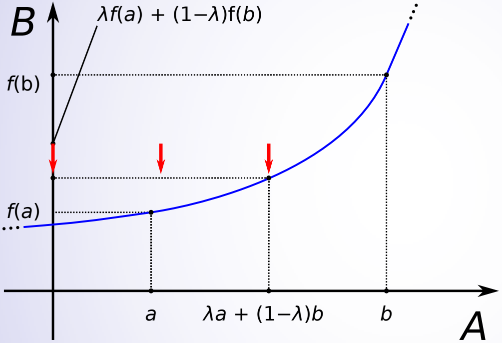

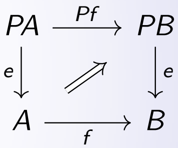

Lax codescent objects

Corollary of [Lack, 2002]:

Let A and B and be P-algebras, and let $f : A \to B$. Then f is concave if and only if $p \mapsto \int f \, dp$ is monotone for $\preceq_\ell$. In other words, if and only if for every $p \preceq_\ell q$,

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Corollary of Hahn-Banach:

Fix now $B = \mathbb{R}$. Let $p, q \in PA$. Then $p \preceq_\ell q$ if and only for every affine monotone map $\tilde{f} : (PA, \preceq_\ell) \to \mathbb{R}$, $\tilde{f}(p) \leq \tilde{f}(q)$.

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Corollary of Hahn-Banach:

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Lax codescent objects

Corollary:

Let A be an unordered P-algebra, let $p, q \in PA$. The following conditions are equivalent:

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This order is known in the literature as the *convex* or *Choquet* order [Winkler, 1985]. The result above is known.

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This order is known in the literature as the *increasing convex order*. The result above, in its full generality, is new.

Acknowledgements

Joint work with Tobias Fritz

Special thanks to Slava Matveev
and Sharwin Rezagholi
(MPI MIS Leipzig)

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