Mix Unitary Categories

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Dagger compact closed categories

Dagger compact closed categories (†-KCC) provide a categorical framework for finite-dimensional quantum mechanics.

The dagger (†) is a contravariant functor which is stationary on objects ($A = A^{\dagger}$) which is an involution ($f^{\dagger\dagger} = f$).

In a †-KCC, quantum processes are represented by completely positive maps.

The CPM construction on a †-KCC chooses the completely positive maps from the category.

FHilb, the category of finite-dimensional Hilbert Spaces and linear maps is the canonical example of a †-KCC.

CPM[FHilb] is precisely the category of "quantum processes."

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Finite versus infinite dimensions

For Hilbert Spaces (and additively enriched categories with negatives) compact closed $\Rightarrow$ finite-dimensionality.

Infinite-dimensional Hilbert spaces have a $\dagger$ but do not have "duals": they are *not* compact closed.

Infinite-dimensional systems occur in many quantum settings including quantum computation and quantum communication.

There have therefore been various attempts to generalize the existing structures and constructions to infinite-dimensions.

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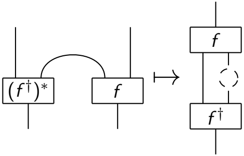
The CP^∞ construction

CP^∞ construction (Coecke and Heunen) generalizes the CPM construction to †-symmetric monoidal categories by reexpressing completely positive maps as follows:



QUESTION: Is there a way to generalize the CPM construction to arbitrary dimensions while retaining duals *and* the dagger?

Linearly distributive categories

$*$-autonomous categories and linearly distributive categories generalize compact closed categories ...

Can quantum ideas be extended in this direction?[1]

They allow for infinite dimensions, have a nice graphical calculus, allow the expression of "duals" ... but what about dagger?

Recall a **linearly distributive category (LDC)** has two monoidal structures $(\otimes, \top, a_\otimes, u_\otimes^L, u_\otimes^R)$ and $(\oplus, \bot, a_\oplus, u_\oplus^L, u_\oplus^R)$ linked by natural transformations called the linear distributors:

$$\partial_L : A \otimes (B \oplus C) \to (A \otimes B) \oplus C$$
$$\partial_R : (A \oplus B) \otimes C \to A \oplus (B \otimes C)$$

[1] See Dusko Pavlovic "Relating Toy Models of Quantum Computation: Comprehension, Complementarity and Dagger Mix Autonomous Categories"

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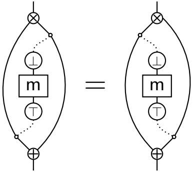
Mix categories

A **mix category** is a LDC with a map $m : \bot \to \top$ in $\mathbb{X}$ such that

$$mx_{A,B} : A \otimes B \to A \oplus B :=$$



mx is called a **mix map**. The mix map is a natural transformation.

It is an **isomix** category if m is an isomorphism.

m being an isomorphism does not make the mx map an isomorphism.

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The Core of mix category

The **core of a mix category**, $\text{Core}(\mathbb{X}) \subseteq \mathbb{X}$, is the full subcategory determined by objects $U \in \mathbb{X}$ for which the natural transformation is also an isomorphism:

$$U \otimes (\_) \xrightarrow{\mathsf{mx}_{U,(\_)}} U \oplus (\_)$$

The core of a mix category is closed to $\otimes$ and $\oplus$.

The core of an isomix category contains the monoidal units $\top$ and $\bot$ and is a **compact LDC** (meaning tensor and par are essentially identical) .

Roadmap

LDC

$\Downarrow$

Define †-LDC

$\Downarrow$

Define unitary isomorphisms in †-LDCs

$\Downarrow$

Generalize CP^{∞} construction for †-LDCs

The †?

The definition of $\dagger : \mathbb{X}^{op} \longrightarrow \mathbb{X}$ as stationary on objects cannot be imported to LDCs because the dagger minimally has to flip the tensor products: $(A \otimes B)^\dagger = A^\dagger \oplus B^\dagger$.

Why? If the dagger is identity-on-objects, then the linear distributor degenerates to an associator:

$$\frac{(\delta_R)^\dagger : (A \oplus (B \otimes C))^\dagger \longrightarrow ((A \oplus B) \otimes C)^\dagger}{(\delta^R)^\dagger : A^\dagger \oplus (B^\dagger \otimes C^\dagger) \longrightarrow (A^\dagger \oplus B^\dagger) \otimes C^\dagger}$$

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†-LDCs

A †-**LDC** is a LDC $\mathbb{X}$ with a dagger functor $\dagger : \mathbb{X}^{op} \longrightarrow \mathbb{X}$ and the natural isomorphisms:

$$
\begin{aligned}
\text{tensor laxors: } & \lambda_\oplus : A^\dagger \oplus B^\dagger \longrightarrow (A \otimes B)^\dagger \\
& \lambda_\otimes : A^\dagger \otimes B^\dagger \longrightarrow (A \oplus B)^\dagger \\
\text{unit laxors: } & \lambda_\top : \top \longrightarrow \bot^\dagger \\
& \lambda_\bot : \bot \longrightarrow \top^\dagger \\
\text{involutor: } & \iota : A \longrightarrow A^{\dagger\dagger}
\end{aligned}
$$

which make † a contravariant (Frobenius) linear equivalence.

Coherences for †-LDCs

Coherences for the interaction between the tensor laxors and the basic natural isomorphisms (6 coherences):

$$
\begin{array}{ccc}
A^\dagger \otimes (B^\dagger \otimes C^\dagger) & \xrightarrow{a_\otimes} & (A^\dagger \otimes B^\dagger) \otimes C^\dagger \\
{\scriptstyle 1 \otimes \lambda_\otimes} \downarrow & & \downarrow {\scriptstyle \lambda_\otimes \otimes 1} \\
(A^\dagger \otimes (B \oplus C)^\dagger) & & (A \oplus B)^\dagger \otimes C^\dagger \\
{\scriptstyle \lambda_\otimes} \downarrow & & \downarrow {\scriptstyle \lambda_\otimes} \\
(A \oplus (B \oplus C))^\dagger & \xrightarrow[(a_\oplus^{-1})^\dagger]{} & ((A \oplus B) \oplus C)^\dagger
\end{array}
$$

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Coherences for †-LDCs (cont.)

Interaction between the unit laxors and the unitors (2 coherences):

$$\begin{array}{ccc} \top \otimes A^{\dagger} & \xrightarrow{\lambda_{\top} \otimes 1} & \bot^{\dagger} \otimes A^{\dagger} \\ {\scriptstyle u^{l}_{\otimes}} \downarrow & & \downarrow {\scriptstyle \lambda_{\otimes}} \\ A^{\dagger} & \xrightarrow[(u^{l}_{\oplus})^{\dagger}]{} & (\bot \oplus A)^{\dagger} \end{array} \qquad \begin{array}{ccc} \bot \oplus A^{\dagger} & \xrightarrow{\lambda_{\bot} \oplus 1} & \top^{\dagger} \oplus A^{\dagger} \\ {\scriptstyle u^{l}_{\oplus}} \downarrow & & \downarrow {\scriptstyle \lambda_{\oplus}} \\ A^{\dagger} & \xrightarrow[(u^{l}_{\otimes})^{\dagger}]{} & (\top \otimes A)^{\dagger} \end{array}$$

Interaction between the involutor and the laxors (4 coherences):

$$\begin{array}{ccc} A \oplus B & \xrightarrow{\iota} & ((A \oplus B)^{\dagger})^{\dagger} \\ {\scriptstyle i \oplus i} \downarrow & & \downarrow {\scriptstyle \lambda_{\otimes}^{\dagger}} \\ (A^{\dagger})^{\dagger} \oplus (B^{\dagger})^{\dagger} & \xrightarrow[\lambda_{\oplus}]{} & (A^{\dagger} \otimes B^{\dagger})^{\dagger} \end{array} \qquad \begin{array}{ccc} \bot & \xrightarrow{\iota} & (\bot^{\dagger})^{\dagger} \\ & {\scriptstyle \lambda_{\bot}} \searrow & \downarrow {\scriptstyle \lambda_{\top}^{\dagger}} \\ & & \top^{\dagger} \end{array}$$

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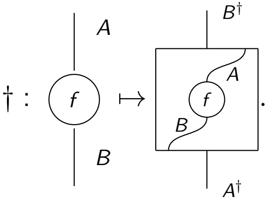
Diagrammatic calculus for †-LDC

Extend the diagrammatic calculus of LDCs

The action of dagger is represented using dagger boxes:



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Isomix †-LDCs

A **mix †-LDC** is a †-LDC with $m : \bot \to \top$ such that:

$$\begin{array}{ccc} \bot & \xrightarrow{\ m\ } & \top \\ {\scriptstyle \lambda_\bot}\downarrow & & \downarrow{\scriptstyle \lambda_\top} \\ \top^\dagger & \xrightarrow[\ m^\dagger\ ]{} & \bot^\dagger \end{array}$$

If m is an isomorphism, then $\mathbb{X}$ is an **isomix †-LDC**.

Lemma: The following diagram commutes in a mix †-LDC:

$$\begin{array}{ccc} A^\dagger \otimes B^\dagger & \xrightarrow{\ mx\ } & A^\dagger \oplus B^\dagger \\ {\scriptstyle \lambda_\otimes}\downarrow & & \downarrow{\scriptstyle \lambda_\oplus} \\ (A \oplus B)^\dagger & \xrightarrow[\ mx^\dagger\ ]{} & (A \otimes B)^\dagger \end{array}$$

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Isomix †-LDCs

Lemma: Suppose $\mathbb{X}$ is a mix †-LDC and $A \in \text{Core}(\mathbb{X})$ then $A^\dagger \in \text{Core}(\mathbb{X})$.

Proof: The natural transformation $A^\dagger \otimes X \xrightarrow{\text{mx}} A^\dagger \oplus X$ is an isomorphism:

$$\begin{array}{ccccc}
A^\dagger \otimes X & \xrightarrow{1 \otimes \iota} & A^\dagger \otimes X^{\dagger\dagger} & \xrightarrow{\lambda_\otimes} & (A \oplus X^\dagger)^\dagger \\
\text{mx} \downarrow & \text{nat. mx} & \text{mx} \downarrow & \text{Lemma above} & \downarrow \text{mx}^\dagger \\
A^\dagger \oplus X & \xrightarrow[1 \oplus \iota]{} & A^\dagger \oplus A^{\dagger\dagger} & \xrightarrow[\lambda_\oplus]{} & (A \otimes X^\dagger)^\dagger
\end{array}$$

commutes.

Next step: Unitary structure

✓ Define †-LDC

? Define unitary isomorphisms

The usual definition of unitary maps

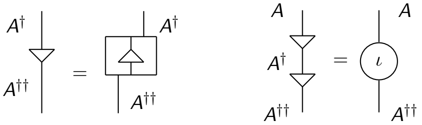
$$(f^{\dagger} : B^{\dagger} \to A^{\dagger} = f^{-1} : B \longrightarrow A)$$

only works when the † functor is stationary on objects.

Unitary structure

An isomix †-LDC has **unitary structure** in case there is a small class of objects called **unitary objects** such that:

- Every unitary object, $A \in \mathcal{U}$, is in the core;
- The dagger of a unitary object is unitary;
- Each unitary object $A \in \mathcal{U}$ comes equipped with an isomorphism, the **unitary structure** of A,  $: A \xrightarrow{\varphi_A} A^\dagger$ such that



$$\varphi_{A^\dagger} = ((\varphi_A)^{-1})^\dagger \qquad (\varphi_A \varphi_{A^\dagger}) = \iota$$

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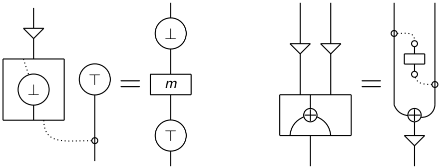
Unitary structure (cont.)

- $\top, \bot$ are unitary objects with:

$$\varphi_\bot = \mathsf{m}\lambda_\top \qquad \varphi_\top = \mathsf{m}^{-1}\lambda_\bot$$

- If A and B are unitary objects then $A \otimes B$ and $A \oplus B$ are unitary objects such that:

$$(\varphi_A \otimes \varphi_B)\lambda_\otimes = \mathsf{mx}\ \varphi_{A \oplus B} : A \otimes B \to (A \otimes B)^\dagger$$
$$\varphi_{A \otimes B}\lambda_\oplus^{-1} = \mathsf{mx}(\varphi_A \oplus \varphi_B) : A \otimes B \to A^\dagger \oplus B^\dagger$$



$\varphi_\bot \lambda_\top^{-1} = \mathsf{m}$ $\qquad$ $(\varphi_A \otimes \varphi_B)\lambda_\otimes = \mathsf{mx}\ \varphi_{A \oplus B}$

Mix Unitary Category (MUC)

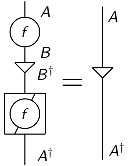
An iso-mix $\dagger$-LDC with unitary structure is a **mix unitary category** (**MUC**).

The unitary objects of a MUC, $\mathbb{X}$, determine a full subcategory, $\text{UCore}(\mathbb{X}) \subseteq \mathbb{X}$, called the **unitary core**. The unitary core is a **unitary category**.

Remark: In order to obtain the right functorial properties a (general) MUC is an isomix $\dagger$-category with a full and faithful structure preserving inclusion of a unitary category.

Unitary isomorphisms

Suppose A and B are unitary objects. An isomorphism $A \xrightarrow{f} B$ is said to be a **unitary isomorphism** if the following diagram commutes:



$$f \varphi_B f^\dagger = \varphi_A$$

Lemma: In a MUC

- $f^\dagger$ is a unitary map iff f is;
- $f \otimes g$ and $f \oplus g$ are unitary maps whenever f and g are.
- $a_\otimes, a_\oplus, c_\otimes, c_\oplus, \delta^L$, m, and mx are unitary isomorphisms.
- $\lambda_\otimes, \lambda_\oplus, \lambda_\top, \lambda_\perp$, and ι are unitary isomorphisms.
- φ_A is a unitary isomorphisms for for all unitary objects A.

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Five examples of MUCs

- **†-KCC** These give a compact closed MUC with a stationary dagger and trivial unitary structure
- **$FFVec_{\mathbb{C}}$** "Framed" vector spaces (vector spaces with a chosen basis) is a compact closed MUC with non-trivial unitary structure.
- **$Fin_{\mathbb{C}}$** $\mathbb{C}$-modules over finiteness spaces is a $*$-autonomous category: maps are infinite dimensional matrices with composition controlled (by types) to avoid infinite sums. The unitary subcategory is just $Mat(\mathbb{C})$.
- **$Bicomp(\mathbb{X})$** The bicompletion of a †-KCC, $\mathbb{X}$ is a mix †-$*$-autonomous category with unitary objects in $\mathbb{X}$.
- **$Chu_{\mathbb{Y}}(I)$** The Chu construction on a symmetric monoidal closed category with conjugation, with dualizing object the unit I, gives a MUC.

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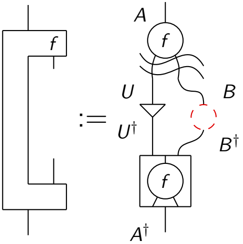
Next step: CP^{∞} construction on MUCs

- ✓ Define †-LDC
- ✓ Define unitary isomorphisms
- ✓ Examples
- ? CP^{∞} construction on MUC

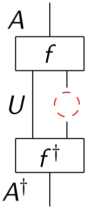
Krauss maps

In a MUC, a map $f : A \longrightarrow U \oplus B$ of $\mathbb{X}$ where U is a unitary object is called a **Krauss map** $f : A \rightarrow_U B$. U is called the **ancillary system** of f.

In a MUC, quantum processes are represented using Krauss maps as follows:



analogous to



in †-SMCs.

$$A \xrightarrow{f} U \oplus B \xrightarrow{\mathrm{mx}^{-1}} U \otimes B \xrightarrow{\varphi \otimes 1} U^\dagger \otimes B$$

$$U^\dagger \otimes B^\dagger \xrightarrow{\lambda_\otimes} (U \oplus B)^\dagger \xrightarrow{f^\dagger} A^\dagger$$

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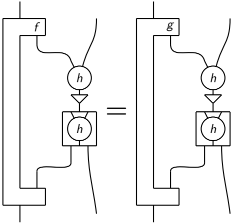
Combinator and test maps

Two Krauss maps $f : A \to_{U_1} B$ and $g : A \to_{U_2} B$ are equivalent, $f \sim g$, if for all test maps $h : B \otimes X \to V$ where V is an unitary object, the following equation holds:



Lemma: Let $f : A \to_{U_1} B$ and $f' : A \to_{U_2} B$ be Krauss maps such that $U_1 \xrightarrow{\alpha} U_2$ is a unitary isomorphism with $f' = (\alpha \oplus 1)f$, then $f \sim f'$. In this case, f is said to be **unitarily isomorphic** to f'.

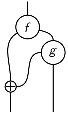
CP^∞ construction

Given a MUC, $\mathbb{X}$, define $\mathrm{CP}^\infty(\mathbb{X})$ to have:

Objects: as of $\mathbb{X}$

Maps:

$$\mathrm{CP}^\infty(\mathbb{X})(A,B) := \{f \in \mathbb{X}(A, U \oplus B) | U \in \mathbb{X} \text{ and } U \text{ is unitary}\}/\sim$$

Composition: 

Identity: $A \xrightarrow{(u_\oplus^L)^{-1}} \bot \oplus A \in \mathbb{X}$

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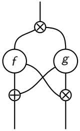
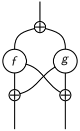
Tensor and Par

$\mathrm{CP}^\infty(\mathbb{X})$ inherits tensor and par from $\mathbb{X}$:

$f \widehat{\otimes} g :=$ 

$f \widehat{\oplus} g :=$ 

$$\widehat{\top} := \top \qquad \widehat{\bot} := \bot$$

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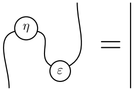
Linear adjoints

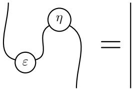
Suppose $\mathbb{X}$ is a LDC and $A, B \in \mathbb{X}$. Then, B is **left linear adjoint** to A $(\eta, \varepsilon) : B \dashv\!\!\vdash A$, if there exists

$$\eta : \top \to B \oplus A \qquad\qquad \varepsilon : A \otimes B \to \bot$$

such that the following triangle equalities hold:

$$\begin{array}{ccccc} B & \xrightarrow{(u_\otimes^L)^{-1}} & \top \otimes B & \xrightarrow{\eta \otimes 1} & (B \oplus A) \otimes B \\ \| & & & & \downarrow{\partial_R} \\ B & \xleftarrow[u_\oplus^R]{} & B \oplus \bot & \xleftarrow[1 \oplus \varepsilon]{} & B \oplus (A \otimes B) \end{array} \qquad \begin{array}{ccccc} A & \xrightarrow{(u_\otimes^R)^{-1}} & A \otimes \top & \xrightarrow{1 \otimes \eta} & A \otimes (B \oplus A) \\ \| & & & & \downarrow{\partial_L} \\ A & \xleftarrow[u_\oplus^L]{} & \bot \oplus A & \xleftarrow[\varepsilon \oplus 1]{} & (A \otimes B) \oplus A \end{array}$$



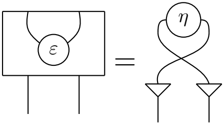


When every object of a MUC has a linear adjoint, it is called a $*$- **MUC**.

Unitary linear adjoints

A **unitary linear adjoint** $(\eta, \varepsilon) : A \dashv\!\!\vdash_u B$ is a linear adjoint, $A \dashv\!\!\vdash B$ with A and B being unitary objects satisfying:

$$\eta_A(\varphi_A \oplus \varphi_B)c_\oplus = \lambda_\top \varepsilon^\dagger \lambda_\oplus^{-1} \qquad (\varphi_A \otimes \varphi_B)\lambda_\otimes \eta_A^\dagger = c_\otimes \varepsilon_A \lambda_\perp$$



$$\lambda_\top \varepsilon^\dagger \lambda_\oplus^{-1} = \eta c_\oplus (\varphi_A \oplus \varphi_B)$$

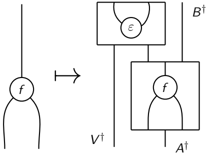
A MUC in which every unitary object has a unitary linear adjoint is called a **MUdC**.

Dagger functor for $\mathrm{CP}^\infty(\mathbb{X})$

Proposition: If $\mathbb{X}$ is a $*$-MUdC, then $\mathrm{CP}^\infty(\mathbb{X})$ is a $*$-MUdC.

Sketch of proof: Suppose $f : A \longrightarrow U \oplus B$ and $(\eta, \varepsilon) : V \dashv\!\!\vdash_u U$

$\dagger : \mathrm{CP}^\infty(\mathbb{X})^{\mathrm{op}} \longrightarrow \mathrm{CP}^\infty(\mathbb{X});$



Unitary structure and unitary linear adjoints are preserved due to the functoriality of Q.

Summary

Generalized one important construction of categorical quantum mechanics to MUCs!

1. Mix Unitary Categories are †-LDCs with unitary subcategory.
2. There is a diagrammatic calculus for MUCs.
3. When unitary objects have unitary linear adjoint, then the unitary core is a dagger compact closed category.
4. CP^{∞} on MUCs generalizes CP^{∞} construction on †-SMCs (auxillary systems in the unitary core).
5. The construction is functorial and produces a *-MUdC when every (unitary) object has a (unitary) linear adjoint.

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