Projections for Hopf quasigroups

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- **MOTIVATION:**
  1. **Radford, D. E.**, The structure of Hopf algebras with a projection, *J. Algebra* **92** (1985) 322-347.
  2. **Majid, S.**, Crossed products by braided groups and bosonization, *J. Algebra* **163** (1994) 165-190.

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- Let $\mathbb{F}$ be a field, $\mathcal{C} = \mathbb{F} - Vect$ and $\otimes$ the tensor product over $\mathbb{F}$. Let H be a Hopf algebra in $\mathcal{C}$ with product $\mu_H(h \otimes g) = hg$, coproduct $\delta_H(h) = h_{(1)} \otimes h_{(2)}$ and antipode λ_H.

  A left-left Yetter-Drinfeld module M over H is simultaneously a left H-module and a left H-comodule, with action and coaction

$$\varphi_M(h \otimes m) = h \bullet m, \quad \rho_M(m) = m_{[1]} \otimes m_{[2]},$$

  satisfying the compatibility condition

$$(h_{(1)} \bullet m)_{[1]} h_{(2)} \otimes (h_{(1)} \bullet m)_{[2]} = h_{(1)} m_{[1]} \otimes h_{(2)} \bullet m_{[2]}.$$

  We denote by ${}^H_H\mathcal{YD}$ the category of left-left Yetter-Drinfeld modules over H. The morphisms in this category preserve both the action and the coaction of H. With the usual tensor product module and comodule structure ${}^H_H\mathcal{YD}$ is monoidal, and, if the antipode is bijective, ${}^H_H\mathcal{YD}$ is braided.

- Let H, B Hopf algebras and $f : H \to B$, $g : B \to H$ Hopf algebra morphisms such that $g \circ f = id_H$ (i.e., (f, g, B) is a Hopf algebra projection over H). If we define the subalgebra of coinvariants by

$$B^{coH} = \{b \in B \ : \ b_{(1)} \otimes g(b_{(2)}) = b \otimes 1_H\}$$

the object B^{coH} is a Hopf algebra in ${}^{H}_{H}\mathcal{YD}$.

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the object B^{coH} is a Hopf algebra in ${}^H_H\mathcal{YD}$.

- Given a Hopf algebra D in ${}^H_H\mathcal{YD}$ with antipode λ_D, it is possible to define a new Hopf algebra in $\mathcal{C}$, called by Majid the bosonization of D, and denoted by

$$D \rtimes H = (D \otimes H, \eta_{D \rtimes H}, \mu_{D \rtimes H}, \varepsilon_{D \rtimes H}, \delta_{D \rtimes H}, \lambda_{D \rtimes H})$$

In this case $\eta_{D \rtimes H}(1_{\mathbb{F}}) = \eta_D(1_{\mathbb{F}}) \otimes \eta_H(1_{\mathbb{F}})$, $\varepsilon_{D \rtimes H}(d \otimes h) = \varepsilon_D(d)\varepsilon_H(h)$, and

$$\mu_{D \rtimes H}(d \otimes h \otimes e \otimes g) = d(h_{(1)} \bullet e) \otimes h_{(2)} g,$$

$$\delta_{D \rtimes H}(d \otimes h) = d_{(1)} \otimes d_{(2)[1]} h_{(1)} \otimes d_{(2)[2]} \otimes h_{(2)},$$

$$\lambda_{D \rtimes H}(d \otimes h) = \lambda_H(d_{[1]} h)_{(1)} \bullet \lambda_D(d_{[2]}) \otimes \lambda_H(d_{[1]} h)_{(2)}.$$

- The morphisms

$$f : H \to D \rtimes H, \;\; f(h) = 1_D \otimes h$$

$$g : D \rtimes H \to H, \;\; g(d \otimes h) = \varepsilon_D(d)h$$

are Hopf algebra morphisms in $\mathcal{C}$ such that $g \circ f = id_H$ and

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- On the other hand, if H, B are Hopf algebras and $f : H \to B$, $g : B \to H$ are Hopf algebra morphisms such that $g \circ f = id_H$, we have

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are Hopf algebra morphisms in $\mathcal{C}$ such that $g \circ f = id_H$ and

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- On the other hand, if H, B are Hopf algebras and $f : H \to B$, $g : B \to H$ are Hopf algebra morphisms such that $g \circ f = id_H$, we have

$$B^{coH} \rtimes H \simeq B$$

as Hopf algebras in $\mathcal{C}$.

Theorem

Let H be a Hopf algebra with bijective antipode. If $Proj(H)$ denotes the category of Hopf algebra projections over H, and $\mathcal{H}({}_H^H\mathcal{YD})$ the category of Hopf algebras in ${}_H^H\mathcal{YD}$, they are equivalent categories.

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1. **Alonso Álvarez J.N., Fernández Vilaboa J.M., González Rodríguez R., Soneira Calvo, C.,** Projections and Yetter-Drinfel'd modules over Hopf (co)quasigroups, *J. Algebra* **443** (2015), 153-199.
2. **Alonso Álvarez, J.N., Fernández Vilaboa, J.M. y González Rodríguez, R.,** Multiplication alteration by two-cocycles. The non-associative version arXiv:1703.01829 (2017).

Outline

Hopf quasigroups

- From now on $\mathcal{C}$ denotes a symmetric monoidal category with tensor product denoted by $\otimes$ and unit object K. With c we will denote the braiding.
  Without loss of generality, by the coherence theorems, we can assume the monoidal structure of $\mathcal{C}$ strict. Then, in this talk, we omit explicitly the associativity and unit constraints.

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  Without loss of generality, by the coherence theorems, we can assume the monoidal structure of $\mathcal{C}$ strict. Then, in this talk, we omit explicitly the associativity and unit constraints.
- We also assume that every idempotent morphism $q : Y \to Y$ in $\mathcal{C}$ splits, i.e., there exist an object Z (image of q) and morphisms $i : Z \to Y$ (injection) and $p : Y \to Z$ (projection) such that $q = i \circ p$ and $p \circ i = id_Z$.

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- For simplicity of notation, given three objects V, U, B in $\mathcal{C}$ and a morphism $f : V \to U$, we write

$$B \otimes f \text{ for } id_B \otimes f \text{ and } f \otimes B \text{ for } f \otimes id_B.$$

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- (A, η_A, μ_A) is a unital magma, i.e. $\eta_A : K \to A$ (unit) and $\mu_A : A \otimes A \to A$ (product) are morphisms in $\mathcal{C}$ such that

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- $(C, \varepsilon_C, \delta_C)$ is a comonoid with comultiplication δ_C and counit ε_C.
- If $f, g : C \to A$ are morphisms, $f * g$ denotes the convolution product.

$$f * g = \mu_A \circ (f \otimes g) \circ \delta_C.$$

Definition

A non-associative bimonoid in the category $\mathcal{C}$ is a unital magma (H, η_H, μ_H) and a comonoid $(H, \varepsilon_H, \delta_H)$ such that ε_H and δ_H are morphisms of unital magmas (equivalently, η_H and μ_H are morphisms of counital comagmas). Then the following identities hold:

$$\varepsilon_H \circ \eta_H = id_K, \quad \varepsilon_H \circ \mu_H = \varepsilon_H \otimes \varepsilon_H,$$

$$\delta_H \circ \eta_H = \eta_H \otimes \eta_H, \quad \delta_H \circ \mu_H = (\mu_H \otimes \mu_H) \circ \delta_{H \otimes H},$$

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Definition

A morphism $f : H \to B$ between non-associative bimonoids H and B is a morphism of unital magmas and comonoids, i.e.,

$$f \circ \eta_H = \eta_B, \quad \mu_B \circ (f \otimes f) = f \circ \mu_H,$$

$$\varepsilon_B \circ f = \varepsilon_H, \quad (f \otimes f) \circ \delta_H = \delta_B \circ f.$$

The above definition is the monoidal version of the notion of Hopf quasigroup (also called non-associative Hopf algebra with the inverse property, or non-associative IP Hopf algebra) introduced in

Klim, J., Majid, S., Hopf quasigroups and the algebraic 7-sphere, *J. Algebra* **323** (2010), 3067-3110.

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A Hopf quasigroup H in $\mathcal{C}$ is a non-associative bimonoid such that there exists a morphism $\lambda_H : H \to H$ in $\mathcal{C}$ (called the antipode of H) satisfying

$$\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) = \varepsilon_H \otimes H = \mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H).$$

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- If H is a Hopf quasigroup in $\mathcal{C}$, the antipode λ_H is unique, antimultiplicative, anticomultiplicative, and leaves the unit and the counit invariant, i.e.,

$$\lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \quad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H,$$

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$$\lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H.$$

- Also, if H is a Hopf quasigroup we have

$$\lambda_H * id_H = \eta_H \otimes \varepsilon_H = id_H * \lambda_H.$$

- A morphism of Hopf quasigroups $f : H \rightarrow B$ is a morphism of non-associative bimonoids. Then
$$\lambda_B \circ f = f \circ \lambda_H.$$

- A morphism of Hopf quasigroups $f : H \to B$ is a morphism of non-associative bimonoids. Then
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- Note that a Hopf quasigroup is associative if an only if is a Hopf monoid.

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Example

A quasigroup is a set Q together with a product such that for any two elements $u, v \in Q$ the equations $ux = v$, $xu = v$ and $uv = x$ have unique solutions in Q. A quasigroup L which contains an element e_L such that $ue_L = u = e_L u$ for every $u \in L$ is called a loop. A loop L is said to be a loop with the inverse property (for brevity an I.P. loop) if and only if, to every element $u \in L$, there corresponds an element $u^{-1} \in L$ such that the equations
$$u^{-1}(uv) = v = (vu)u^{-1}$$
hold for every $v \in L$.
If L is an I.P. loop, it is easy to show that for all $u \in L$ the element u^{-1} is unique and
$$u^{-1}u = e_L = uu^{-1}.$$
Moreover, the mapping $u \rightarrow u^{-1}$ is an anti-automorphism of the I.P. loop L:
$$(uv)^{-1} = v^{-1}u^{-1}$$

Let R be a commutative ring and let L be an I.P. loop. Then,

$$RL = \bigoplus_{u \in L} Ru$$

is a cocommutative Hopf quasigroup with product defined by the linear extension of the one defined in L and

$$\delta_{RL}(u) = u \otimes u, \;\; \varepsilon_{RL}(u) = 1_R, \;\; \lambda_{RL}(u) = u^{-1}$$

on the basis elements. Note that, in this case, λ_{RL} is an isomorphism and $\lambda_{RL} \circ \lambda_{RL} = id_{RL}$.

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Example

The enveloping algebra $U(L)$ of a Malcev algebra L, introduced in

> **Pérez-Izquierdo, J.M., Shestakov, I.P.**, An envelope for Malcev algebras, *J. Algebra* **272** (2004), 379-393,

when the groundfield has characteristic not 2, 3 is an example of cocommutative Hopf quasigroup.

Yetter-Drinfeld modules and projections

Definition

Let H be a Hopf quasigroup. We say that $M = (M, \varphi_M, \rho_M)$ is a left-left Yetter-Drinfeld module over H if which satisfies the following equalities:

(a1) (M, φ_M) is a left H-module, i.e.,

$$\varphi_M \circ (\eta_H \otimes M) = id_M, \ \ \varphi_M \circ (\varphi_M \otimes M) = \varphi_M \circ (\mu_H \otimes M).$$

(a2) (M, ρ_M) is a left H-comodule, i.e.,

$$(\varepsilon_H \otimes M) \circ \rho_M = id_M, \ \ (\rho_M \otimes M) \circ \rho_M = (\delta_H \otimes M) \circ \rho_M.$$

(a3) $(\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ ((\rho_M \circ \varphi_M) \otimes H) \circ (H \otimes c_{H,M}) \circ (\delta_H \otimes M)$
$= (\mu_H \otimes \varphi_M) \circ (H \otimes c_{H,H} \otimes M) \circ (\delta_H \otimes \rho_M)$.

(a4) $(\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ (\rho_M \otimes \mu_H)$
$= (\mu_H \otimes M) \circ (\mu_H \otimes c_{M,H}) \circ (H \otimes c_{M,H} \otimes H) \circ (\rho_M \otimes H \otimes H)$.

(a5) $(\mu_H \otimes M) \circ (H \otimes \mu_H \otimes M) \circ (H \otimes H \otimes c_{M,H}) \circ (H \otimes \rho_M \otimes H)$
$= (\mu_H \otimes M) \circ (\mu_H \otimes c_{M,H}) \circ (H \otimes \rho_M \otimes H)$.

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(a1) (M, φ_M) is a left H-module, i.e.,

$$\varphi_M \circ (\eta_H \otimes M) = id_M, \;\; \varphi_M \circ (\varphi_M \otimes M) = \varphi_M \circ (\mu_H \otimes M).$$

(a2) (M, ρ_M) is a left H-comodule, i.e.,

$$(\varepsilon_H \otimes M) \circ \rho_M = id_M, \;\; (\rho_M \otimes M) \circ \rho_M = (\delta_H \otimes M) \circ \rho_M.$$

(a3) $(\mu_H \otimes M) \circ (H \otimes c_{M,H}) \circ ((\rho_M \circ \varphi_M) \otimes H) \circ (H \otimes c_{H,M}) \circ (\delta_H \otimes M)$
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$= (\mu_H \otimes M) \circ (\mu_H \otimes c_{M,H}) \circ (H \otimes \rho_M \otimes H)$.

Let M and N be two left-left Yetter-Drinfeld modules over H. We say that $f : M \to N$ is a morphism of left-left Yetter-Drinfeld modules if f is a morphism of H-modules and H-comodules.

We denote by ${}^{H}_{H}\mathcal{YD}$ the category of left-left Yetter-Drinfeld modules over H. Note that if H is a Hopf monoid, conditions (a4) and (a5) trivialize. In this case, ${}^{H}_{H}\mathcal{YD}$ is the classical category of left-left Yetter-Drinfeld modules over H.

We denote by ${}_{H}^{H}\mathcal{YD}$ the category of left-left Yetter-Drinfeld modules over H. Note that if H is a Hopf monoid, conditions (a4) and (a5) trivialize. In this case, ${}_{H}^{H}\mathcal{YD}$ is the classical category of left-left Yetter-Drinfeld modules over H.

Let (M, φ_M, ρ_M) and (N, φ_N, ρ_N) two objects in ${}_{H}^{H}\mathcal{YD}$. Then $M \otimes N$, with the diagonal structure $\varphi_{M\otimes N}$ and the codiagonal costructure $\rho_{M\otimes N}$, is an object in ${}_{H}^{H}\mathcal{YD}$. Then $({}_{H}^{H}\mathcal{YD}, \otimes, K)$ is a strict monoidal category. If moreover λ_H is an isomorphism, $({}_{H}^{H}\mathcal{YD}, \otimes, K)$ is a strict braided monoidal category where the braiding t and its inverse are defined by

$$t_{M,N} = (\varphi_N \otimes M) \circ (H \otimes c_{M,N}) \circ (\rho_M \otimes N)$$

and

$$t_{M,N}^{-1} = c_{N,M} \circ ((\varphi_N \circ c_{N,H}) \otimes M) \circ (N \otimes \lambda_H^{-1} \otimes M) \circ (N \otimes \rho_M),$$

respectively

Definition

Let H be a Hopf quasigroup such that its antipode is an isomorphism. Let (D, u_D, m_D) be a unital magma in $\mathcal{C}$ such that (D, e_D, Δ_D) is a comonoid in $\mathcal{C}$, and let $s_D : D \to D$ be a morphism in $\mathcal{C}$. We say that the triple (D, φ_D, ρ_D) is a Hopf quasigroup in ${}^{H}_{H}\mathcal{YD}$ if:

(b1) The triple (D, φ_D, ρ_D) is a left-left Yetter-Drinfeld H-module.

(b2) The triple (D, u_D, m_D) is a unital magma in ${}^{H}_{H}\mathcal{YD}$.

(b3) The triple (D, e_D, Δ_D) is a a comonoid in ${}^{H}_{H}\mathcal{YD}$.

(b4) The following identities hold:

- (b4-1) $e_D \circ u_D = id_K$,
- (b4-2) $e_D \circ m_D = e_D \otimes e_D$,
- (b4-3) $\Delta_D \circ e_D = e_D \otimes e_D$,
- (b4-4) $\Delta_D \circ m_D = (m_D \otimes m_D) \circ (D \otimes t_{D,D} \otimes D) \circ (\Delta_D \otimes \Delta_D)$,

where $t_{D,D}$ is the braiding of ${}^{H}_{H}\mathcal{YD}$.

(b5) The following identities hold:

- (b5-1) $m_D \circ (s_D \otimes m_D) \circ (\Delta_D \otimes D) = e_D \otimes D = m_D \circ (D \otimes m_D) \circ (D \otimes s_D \otimes D) \circ (\Delta_D \otimes D)$.
- (b5-2) $m_D \circ (m_D \otimes D) \circ (D \otimes s_D \otimes D) \circ (D \otimes \Delta_D) = D \otimes e_D = \mu_D \circ (m_D \otimes s_D) \circ (D \otimes \Delta_D)$.

Note that under these conditions, s_D is a morphism in ${}_H^H\mathcal{YD}$.

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Theorem

Let H be a Hofpf quasigroup such that λ_H is an isomorphism. If $(D, \varphi_D, \varrho_D)$ is a Hopf quasigroup in ${}_H^H\mathcal{YD}$, then

$$D \rtimes H = (D \otimes H, \eta_{D\rtimes H}, \mu_{D\rtimes H}, \varepsilon_{D\rtimes H}, \delta_{D\rtimes H}, \lambda_{D\rtimes H})$$

is a Hopf quasigroup in $\mathcal{C}$ (the bosonization of D), with the biproduct structure induced by the smash product coproduct, i.e.,

$$\eta_{D\rtimes H} = \eta_D \otimes \eta_H, \quad \mu_{D\rtimes H} = (\mu_D \otimes \mu_H) \circ (D \otimes \Psi_D^H \otimes H),$$

$$\varepsilon_{D\rtimes H} = \varepsilon_D \otimes \varepsilon_H, \quad \delta_{D\rtimes H} = (D \otimes \Gamma_D^H \otimes H) \circ (\delta_D \otimes \delta_H),$$

$$\lambda_{D\rtimes H} = \Psi_D^H \circ (\lambda_H \otimes \lambda_D) \circ \Gamma_D^H,$$

where the morphisms $\Psi_D^H : H \otimes D \to D \otimes H$, $\Gamma_D^H : D \otimes H \to H \otimes D$, are defined by

$$\Psi_D^H = (\varphi_D \otimes H) \circ (H \otimes c_{H,D}) \circ (\delta_H \otimes D), \quad \Gamma_D^H = (\mu_H \otimes D) \circ (H \otimes c_{D,H}) \circ (\rho_D \otimes H).$$

Proposition

Let H and B be Hopf quasigroups and let $f : H \to B$ and $g : B \to H$ be morphisms of Hopf quasigroups such that $g \circ f = id_H$. Then

$$q_H^B = id_B * (f \circ \lambda_H \circ g) : B \to B$$

is an idempotent morphism. Moreover, if B^{coH} is the image of q_H^B and $p_H^B : B \to B^{coH}$, $i_H^B : B^{coH} \to B$ a factorization of q_H^B,

$$B^{coH} \xrightarrow{\; i_H^B \;} B \overset{(B \otimes g) \circ \delta_B}{\underset{B \otimes \eta_H}{\rightrightarrows}} B \otimes H$$

is an equalizer diagram. As a consequence, the triple $(B^{coH}, u_{B^{coH}}, m_{B^{coH}})$ is a unital magma where $u_{B^{coH}}$ and $m_{B^{coH}}$ are the factorizations, through the equalizer i_H^B, of the morphisms η_B and $\mu_B \circ (i_H^B \otimes i_H^B)$, respectively.

Definition

Let H be a Hopf quasigroup. A Hopf quasigroup projection over H is a triple (B, f, g) where B is a Hopf quasigroup, $f : H \to B$ and $g : B \to H$ are morphisms of Hopf quasigroups such that $g \circ f = id_H$, and for the morphisms $q_H^B = id_B * (f \circ \lambda_H \circ g)$ the equality

$$q_H^B \circ \mu_B \otimes (B \otimes q_H^B) = q_H^B \circ \mu_B \quad (1)$$

holds.

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If B is a Hopf monoid the identity (1) always holds. Then, in the associative setting, the previous definition is the definition of Hopf monoid projection over H.

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If B is a Hopf monoid the identity (1) always holds. Then, in the associative setting, the previous definition is the definition of Hopf monoid projection over H.

A morphism between two Hopf quasigroup projections (B, f, g) and (B', f', g') over H is a Hopf quasigroup morphism $h : B \to B'$ such that $h \circ f = f'$, $g' \circ h = g$.
Hopf quasigroup projections over H and morphisms of Hopf quasigroup projections with the obvious composition form a category, denoted by

$$\mathcal{P}roj(H).$$

Proposition

If (B, f, g) is a Hopf quasigroup projection over H,

$$B \otimes H \underset{B \otimes \varepsilon_H}{\overset{\mu_B \circ (B \otimes f)}{\rightrightarrows}} B \xrightarrow{p_H^B} B^{coH}$$

is a coequalizer diagram. Then, the triple $(B^{coH}, e_{B^{coH}}, \Delta_{B^{coH}})$ is a comonoid, where $e_{B^{coH}}$ and $\Delta_{B^{coH}}$ are the factorizations, through the coequalizer p_H^B, of the morphisms ε_B and $(p_H^B \otimes p_H^B) \circ \delta_B$, respectively.

Definition

Let H be a Hopf quasigroup. We say that a Hopf quasigroup projection (B, f, g) over H is strong if it satisfies

$$p_H^B \circ \mu_B \circ (B \otimes \mu_B) \circ (i_H^B \otimes f \otimes i_H^B) = p_H^B \circ \mu_B \circ (\mu_B \otimes B) \circ (i_H^B \otimes f \otimes i_H^B),$$

$$p_H^B \circ \mu_B \circ (B \otimes \mu_B) \circ (f \otimes i_H^B \otimes i_H^B) = p_H^B \circ \mu_B \circ (\mu_B \otimes B) \circ (f \otimes i_H^B \otimes i_H^B),$$

$$p_H^B \circ \mu_B \circ (B \otimes \mu_B) \circ (f \otimes f \otimes i_H^B) = p_H^B \circ \mu_B \circ (\mu_B \otimes B) \circ (f \otimes f \otimes i_H^B).$$

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$$p_H^B \circ \mu_B \circ (B \otimes \mu_B) \circ (f \otimes i_H^B \otimes i_H^B) = p_H^B \circ \mu_B \circ (\mu_B \otimes B) \circ (f \otimes i_H^B \otimes i_H^B),$$

$$p_H^B \circ \mu_B \circ (B \otimes \mu_B) \circ (f \otimes f \otimes i_H^B) = p_H^B \circ \mu_B \circ (\mu_B \otimes B) \circ (f \otimes f \otimes i_H^B).$$

If B is a Hopf monoid the previous definition is the definition of Hopf monoid projection over H because the product is associative.

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$$p_H^B \circ \mu_B \circ (B \otimes \mu_B) \circ (f \otimes f \otimes i_H^B) = p_H^B \circ \mu_B \circ (\mu_B \otimes B) \circ (f \otimes f \otimes i_H^B).$$

If B is a Hopf monoid the previous definition is the definition of Hopf monoid projection over H because the product is associative.

Example

Let H be a Hopf quasigroup with invertible antipode. If D is a Hopf quasigroup in ${}_H^H\mathcal{YD}$, the triple $(D \rtimes H, f = \eta_D \otimes H, g = \varepsilon_D \otimes H)$ is a strong Hopf quasigroup projection over H. In this case $q_H^{D\rtimes H} = D \otimes \eta_H \otimes \varepsilon_H$. As a consequence,

$$p_H^{D\rtimes H} = D \otimes \varepsilon_H, \quad i_H^{D\rtimes H} = D \otimes \eta_H$$

and then $(D \rtimes H)^{coH} = D$.

We will denote by

$$\mathcal{SP}roj(H)$$

the category of strong Hopf quasigroup projections over H. The morphisms of $\mathcal{SP}roj(H)$ are the morphisms of $\mathcal{P}roj(H)$.

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Proposition

Let H be a Hopf quasigroup with invertible antipode. If (B, f, g) is a strong Hopf quasigroup projection over H, the triple $(B^{coH}, \varphi_{B^{coH}}, \rho_{B^{coH}})$ is a Hopf quasigroup in ${}^{H}_{H}\mathcal{YD}$, where

$$\varphi_{B^{coH}} = p_H^B \circ \mu_B \circ (f \otimes i_H^B), \quad \rho_{B^{coH}} = (g \otimes p_H^B) \circ \delta_B \circ i_H^B,$$

and $s_{B^{coH}} = p_H^B \circ ((f \circ g) * \lambda_B) \circ i_H^B$.
Moreover,

$$w = \mu_B \circ (i_H^B \otimes f) : B^{coH} \rtimes H \to B$$

is an isomorphism of Hopf quasigroups in $\mathcal{C}$ with inverse

$$w^{-1} = (p_H^B \otimes g) \circ \delta_B.$$

Theorem

Let H be a Hopf quasigroup in $\mathcal{C}$ with invertible antipode. The categories $\mathcal{SP}roj(H)$ and the category of Hopf quasigroups in ${}_H^H\mathcal{YD}$ are equivalent.

Two cocycles and skew pairings for Hopf quasigroups

Definition

Let H be a non-associative bimonoid, and let $\sigma : H \otimes H \to K$ be a convolution invertible morphism. We say that σ is a 2-cocycle if the equality

$$\partial^1(\sigma) * \partial^3(\sigma) = \partial^4(\sigma) * \partial^2(\sigma)$$

holds, where $\partial^1(\sigma) = \varepsilon_H \otimes \sigma$, $\partial^2(\sigma) = \sigma \circ (\mu_H \otimes H)$, $\partial^3(\sigma) = \sigma \circ (H \otimes \mu_H)$ and $\partial^4(\sigma) = \sigma \otimes \varepsilon_H$.

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Definition

A 2-cocycle σ is called normal if further

$$\sigma \circ (\eta_H \otimes H) = \varepsilon_H = \sigma \circ (H \otimes \eta_H),$$

and it is easy to see that if σ is normal so is σ^{-1}.

Proposition

Let H be a non-associative bimonoid. Let σ be a normal 2-cocycle. Define the product μ_{H^σ} as

$$\mu_{H^\sigma} = (\sigma \otimes \mu_H \otimes \sigma^{-1}) \circ (H \otimes H \otimes \delta_{H\otimes H}) \circ \delta_{H\otimes H}.$$

Then $H^\sigma = (H, \eta_{H^\sigma} = \eta_H, \mu_{H^\sigma}, \varepsilon_{H^\sigma} = \varepsilon_H, \delta_{H^\sigma} = \delta_H)$ is a non-associative bimonoid.

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Proposition

Let H be a Hopf quasigroup with antipode λ_H. Let σ be a normal 2-cocycle. Then the non-associative bimonoid H^σ, is a Hopf quasigroup with antipode

$$\lambda_{H^\sigma} = (f \otimes \lambda_H \otimes f^{-1}) \circ (H \otimes \delta_H) \circ \delta_H,$$

where

$$f = \sigma \circ (H \otimes \lambda_H) \circ \delta_H, \quad f^{-1} = \sigma^{-1} \circ (\lambda_H \otimes H) \circ \delta_H.$$

Definition

Let A and H be non-associative bimonoids in $\mathcal{C}$. A skew pairing between A and H over K is a morphism $\tau : A\otimes H \to K$ such that the equalities

(c1) $\tau\circ(\mu_A\otimes H) = (\tau\otimes\tau)\circ(A\otimes c_{A,H}\otimes H)\circ(A\otimes A\otimes\delta_H)$,

(c2) $\tau\circ(A\otimes\mu_H) = (\tau\otimes\tau)\circ(A\otimes c_{A,H}\otimes H)\circ((c_{A,A}\circ\delta_A)\otimes H\otimes H)$,

(c3) $\tau\circ(A\otimes\eta_H) = \varepsilon_A$,

(c4) $\tau\circ(\eta_A\otimes H) = \varepsilon_H$,

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(c3) $\tau \circ (A \otimes \eta_H) = \varepsilon_A$,

(c4) $\tau \circ (\eta_A \otimes H) = \varepsilon_H$,

hold.

Proposition

Let A, H be Hopf quasigroups with antipodes λ_A and λ_H respectively. Let $\tau : A \otimes H \to K$ be a skew pairing. Then τ is convolution invertible with inverse $\tau^{-1} = \tau \circ (\lambda_A \otimes H)$. Moreover, the equalities

$$\tau^{-1} \circ (\eta_A \otimes H) = \varepsilon_H, \quad \tau^{-1} \circ (A \otimes \eta_H) = \varepsilon_A$$

and

$$\tau^{-1} \circ (A \otimes \mu_H) = (\tau^{-1} \otimes \tau^{-1}) \circ (A \otimes c_{A,H} \otimes H) \circ (\delta_A \otimes H \otimes H)$$

hold.

Proposition

Let A, H be Hopf quasigroups with antipodes λ_A, λ_H respectively. Then

$$A \otimes H = (A \otimes H, \eta_{A\otimes H}, \mu_{A\otimes H}, \varepsilon_{A\otimes H}, \delta_{A\otimes H})$$

$$\eta_{A\otimes H} = \eta_A \otimes \eta_H, \;\; \mu_{A\otimes H} = (\mu_A \otimes \mu_H) \circ (A \otimes c_{H,A} \otimes H),$$

$$\varepsilon_{A\otimes H} = \varepsilon_A \otimes \varepsilon_H, \;\; \delta_{A\otimes H} = (A \otimes c_{A,H} \otimes H) \circ (\delta_A \otimes \delta_H),$$

is a Hopf quasigroup with antipode $\lambda_{A\otimes H} = \lambda_A \otimes \lambda_H$.
Moreover, let $\tau : A \otimes H \to K$ be a skew pairing. The morphism

$$\omega = \varepsilon_A \otimes (\tau \circ c_{H,A}) \otimes \varepsilon_H$$

is a normal 2-cocycle with convolution inverse $\omega^{-1} = \varepsilon_A \otimes (\tau^{-1} \circ c_{H,A}) \otimes \varepsilon_H$.

Proposition

Let A, H be Hopf quasigroups with antipodes λ_A, λ_H respectively. Then

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$$\eta_{A\otimes H} = \eta_A \otimes \eta_H, \ \ \mu_{A\otimes H} = (\mu_A \otimes \mu_H) \circ (A \otimes c_{H,A} \otimes H),$$

$$\varepsilon_{A\otimes H} = \varepsilon_A \otimes \varepsilon_H, \ \ \delta_{A\otimes H} = (A \otimes c_{A,H} \otimes H) \circ (\delta_A \otimes \delta_H),$$

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Corollary

Let A, H be Hopf quasigroups with antipodes λ_A, λ_H respectively. Let $\tau : A \otimes H \to K$ be a skew pairing. Then

$$A \bowtie_\tau H = (A \otimes H)^\omega$$

has a structure of Hopf quasigroup.

$$A \bowtie_{\tau} H = (A \otimes H, \eta_{A\bowtie_{\tau} H}, \mu_{A\bowtie_{\tau} H}, \varepsilon_{A\bowtie_{\tau} H}, \delta_{A\bowtie_{\tau} H}, \lambda_{A\bowtie_{\tau} H})$$

$$\eta_{A\bowtie_{\tau} H} = \eta_{A\otimes H},$$

$$\mu_{A\bowtie_{\tau} H} = (\mu_A \otimes \mu_H) \circ (A \otimes \tau \otimes A \otimes H \otimes \tau^{-1} \otimes H)$$

$$\circ (A \otimes \delta_{A\otimes H} \otimes A \otimes H \otimes H) \circ (A \otimes \delta_{A\otimes H} \otimes H) \circ (A \otimes c_{H,A} \otimes H),$$

$$\varepsilon_{A\bowtie_{\tau} H} = \varepsilon_{A\otimes H}, \quad \delta_{A\bowtie_{\tau} H} = \delta_{A\otimes H}$$

and

$$\lambda_{A\bowtie_{\tau} H} = (\tau^{-1} \otimes \lambda_A \otimes \lambda_H \otimes \tau) \circ (A \otimes H \otimes \delta_{A\otimes H}) \circ \delta_{A\otimes H}.$$

Example

Let $\mathbb{F}$ be a field such that Char($\mathbb{F}$)$\neq$ 2 and denote the tensor product over $\mathbb{F}$ as $\otimes$. Consider the nonabelian group $S_3 = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ where σ_0 is the identity, $o(\sigma_1) = o(\sigma_2) = o(\sigma_3) = 2$ and $o(\sigma_4) = o(\sigma_5) = 3$. Let u be an additional element such that $u^2 = 1$. By

> **Chein O.**, Moufang loops of small order I, *Trans. Amer. Math. Soc.* **188** (1974), 31-51.

the set

$$L = M(S_3, 2) = \{\sigma_i u^\alpha \ ; \ \alpha = 0, 1\}$$

is an IP-loop where the product is defined by

$$\sigma_i u^\alpha . \ \sigma_j u^\beta = (\sigma_i^\nu \sigma_j^\mu)^\nu u^{\alpha+\beta}, \ \ \nu = (-1)^\beta, \ \mu = (-1)^{\alpha+\beta}.$$

Then, $A = \mathbb{F}L$ is a cocommutative Hopf quasigroup.

Let H_4 be the 4-dimensional Taft Hopf algebra. This Hopf algebra is the smallest non commutative, non cocommutative Hopf algebra. The basis of H_4 is $\{1, x, y, w = xy\}$ and the multiplication table is defined by

	x	y	w
x	1	w	y
y	$-w$	0	0
w	$-y$	0	0

The costructure of H_4 is given by

$$\delta_{H_4}(x) = x \otimes x,\ \delta_{H_4}(y) = y \otimes x + 1 \otimes y,\ \delta_{H_4}(w) = w \otimes 1 + x \otimes w,$$

$$\varepsilon_{H_4}(x) = 1_{\mathbb{F}},\ \varepsilon_{H_4}(y) = \varepsilon_{H_4}(w) = 0,$$

and the antipode λ_{H_4} is described by

$$\lambda_{H_4}(x) = x,\ \lambda_{H_4}(y) = w,\ \lambda_{H_4}(w) = -y.$$

Then,

$$A \otimes H_4$$

is a non commutative, non cocommutative Hopf quasigroup and the morphism

$$\tau : A \otimes H_4 \to \mathbb{F}$$

defined by

$$\tau(\sigma_i u^\alpha \otimes z) = \begin{cases} 1 & \text{if} \quad z = 1 \\ (-1)^\alpha & \text{if} \quad z = x \\ 0 & \text{if} \quad z = y, w \end{cases}$$

is a skew pairing. Then,

$$\omega = \varepsilon_A \otimes (\tau \circ c_{H_4, A}) \otimes \varepsilon_{H_4}$$

is an invertible normal 2-cocycle. Finally,

$$A \bowtie_\tau H_4$$

is Hopf quasigroup defined by $(A \otimes H_4)^\omega$.

Quasitriangular Hopf quasigroups, skew pairings and projections

Definition

Let H be a Hopf quasigroup. We will say that H is quasitriangular if there exists a morphism $R : K \to H \otimes H$ such that:

(d1) $(\delta_H \otimes H) \circ R = (H \otimes H \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes H) \circ (R \otimes R)$,

(d2) $(H \otimes \delta_H) \circ R = (\mu_H \otimes c_{H,H}) \circ (H \otimes c_{H,H} \otimes H) \circ (R \otimes R)$,

(d3) $\mu_{H\otimes H} \circ ((c_{H,H} \circ \delta_H) \otimes R) = \mu_{H\otimes H} \circ (R \otimes \delta_H)$,

(d4) $(\varepsilon_H \otimes H) \circ R = (H \otimes \varepsilon_H) \circ R = \eta_H$.

Theorem

Let A, H be Hopf quasigroups and let $\tau : A \otimes H \to K$ be a skew pairing. Assume that H is quasitriangular with morphism R. Let $A \bowtie_\tau H$ be the Hopf quasigroup associated to τ and let $g : A \bowtie_\tau H \to H$ be the morphism defined by

$$g = (\tau \otimes \mu_H) \circ (A \otimes R \otimes H).$$

If the following equalities hold

$$\mu_H \circ (g \otimes H) = g \circ (A \otimes \mu_H), \tag{2}$$

$$\mu_H \circ (H \otimes g) = \mu_H \circ (\mu_H \circ H) \circ (H \otimes ((\tau \otimes H) \circ (A \otimes R)) \otimes H), \tag{3}$$

the triple

$$(A \bowtie_\tau H, f, g),$$

where $f = \eta_A \otimes H$ is a strong Hopf quasigroup projection over H.

Theorem

Let A, H be Hopf quasigroups and let $\tau : A \otimes H \to K$ be a skew pairing. Assume that H is quasitriangular with morphism R. Let $A \bowtie_\tau H$ be the Hopf quasigroup associated to τ and let $g : A \bowtie_\tau H \to H$ be the morphism defined by

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If the following equalities hold

$$\mu_H \circ (g \otimes H) = g \circ (A \otimes \mu_H), \tag{2}$$

$$\mu_H \circ (H \otimes g) = \mu_H \circ (\mu_H \circ H) \circ (H \otimes ((\tau \otimes H) \circ (A \otimes R)) \otimes H), \tag{3}$$

the triple

$$(A \bowtie_\tau H, f, g),$$

where $f = \eta_A \otimes H$ is a strong Hopf quasigroup projection over H.

Note that, if H is a Hopf monoid (2) and (3) always hold.

Theorem

Let H be a Hopf quasigroup with invertible antipode. In the conditions of the previous theorem,

$$A = (A \bowtie_{\tau} H)^{coH}$$

and, as a consequence, there exist an action φ_A and a coaction ρ_A such that (A, φ_A, ρ_A) is a Hopf quasigroup in ${}^{H}_{H}\mathcal{YD}$. Moreover,

$$A \bowtie_{\tau} H \cong A \rtimes H$$

as Hopf quasigroups in $\mathcal{C}$.

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$$A \bowtie_{\tau} H \cong A \rtimes H$$

as Hopf quasigroups in $\mathcal{C}$.

Example

Consider the Hopf quasigroup $A \bowtie_{\tau} H_4$ constructed previously wit $A = \mathbb{F}M(S_3, 2)$. The Hopf algebra H_4 is quasitriangular. Therefore, A admits a structure of Hopf quasigroup in the category ${}^{H_4}_{H_4}\mathcal{YD}$. Moreover,

$$A \bowtie_{\tau} H_4 \cong A \rtimes H_4.$$

Theorem

Let H be a Hopf quasigroup with invertible antipode. In the conditions of the previous theorem,

$$A = (A \bowtie_{\tau} H)^{coH}$$

and, as a consequence, there exist an action φ_A and a coaction ρ_A such that (A, φ_A, ρ_A) is a Hopf quasigroup in ${}_H^H\mathcal{YD}$. Moreover,

$$A \bowtie_{\tau} H \cong A \rtimes H$$

as Hopf quasigroups in $\mathcal{C}$.

Example

Consider the Hopf quasigroup $A \bowtie_{\tau} H_4$ constructed previously wit $A = \mathbb{F}M(S_3, 2)$. The Hopf algebra H_4 is quasitriangular. Therefore, A admits a structure of Hopf quasigroup in the category ${}_{H_4}^{H_4}\mathcal{YD}$. Moreover,

$$A \bowtie_{\tau} H_4 \cong A \rtimes H_4.$$

Thank you