The Simpson conjecture (for regular compositions)

Simon Henry

(Masaryk University, Brno)

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... approximately 220 pages in total.

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Simpson's suggestion: just follow Kapranov and Voevodsky's strategy.

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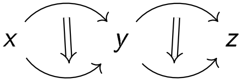
- One wants to generalize this to higher dimension.

- A 2-arrow in $\pi_\infty(X)$ should look like:



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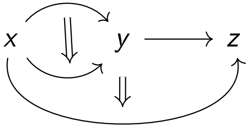

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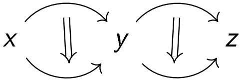

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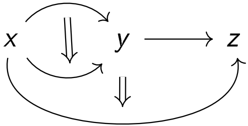

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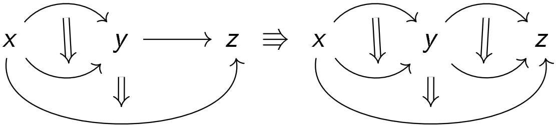

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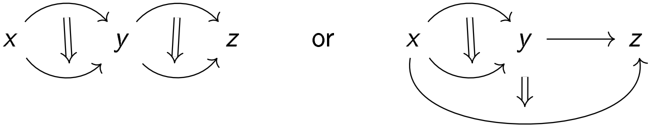

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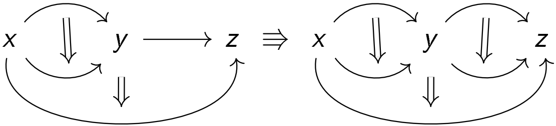

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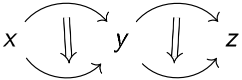

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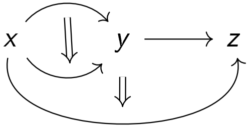

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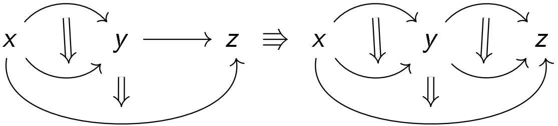

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- They claim to construct something like a model category structure on the category of strict ∞-categories and on the category of presheaves over their category of diagrams.
- Fibrant objects among ∞-categories are those where all arrows are invertible.
- They prove that a natural adjonction $Psh(Diag) \rightleftarrows \infty\text{-cat}$ is a Quillen equivalence.

They use the usual geometric realization of presheaves to construct a Quillen equivalence between $Psh(Diag) \simeq Spaces$. For each presheaf $X \in Psh(Diag)$ on defines its geometric realization as:

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$$\boxed{Spaces \overset{\sim}{\leftarrow} Psh(Diag) \overset{\sim}{\rightarrow} \infty - Cat}$$

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But it does not work: it is not possible to produce a notion of diagram constructed this way in general (because of the Eckmann-Hilton argument).

Theorem (H. 1711.00744)

Such a notion of diagrams exists if one restrict to "non-unital ∞-category". i.e. one only consider diagram where each arrow of dimension n has source and targets of dimension $n-1$ exactly.

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Theorem (H. 1711.00744)

The category of "positive" or "non-unital" polygraphs is equivalent to the category $Psh(Plex)$.

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- One construct such weak model structures on $Psh(Plex)$ and on (Non-unital ∞-Cat) which makes them Quillen equivalent.

Question:

$$Psh(Plex) \overset{??}{\simeq} Space$$

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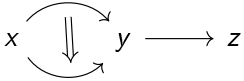

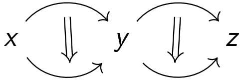

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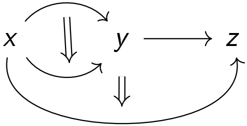

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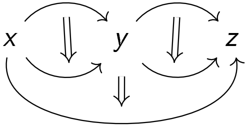

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In the regular framework, this problem of "non-contractible plexes" disapear, and one can finish the proof to get two Quillen equivalences:

$$Spaces \overset{\sim}{\leftrightarrows} Psh(Regular - Plex) \overset{\sim}{\rightleftarrows} (\text{“Regular” } \infty\text{-categories})$$