Braided skew monoidal categories

Stephen Lack
Macquarie University

joint work with John Bourke

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References

- Szlachanyi (2012): *Skew monoidal categories and bialgebroids*
- Street (2013): *Skew-closed categories*
- Lack-Street (2012–): 5 papers so far on skew monoidal categories
- Bourke (2017): *Skew structures in 2-category theory and homotopy theory*
- Bourke-Lack (2018–): 3 papers so far ...

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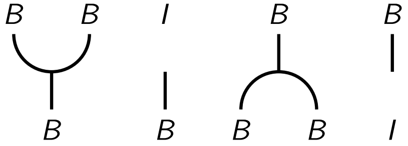
Examples

- (CT2013) From quantum algebra (bialgebras, bialgebroids, ...)
- (CT2015) From 2-category theory (2-categories of categoriess with "commutative" algebraic structure)
- (CT2014) Other (operadic categories)

Quantum examples

B bialgebra in a braided monoidal category $\mathcal{V}$.



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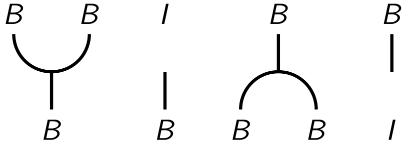
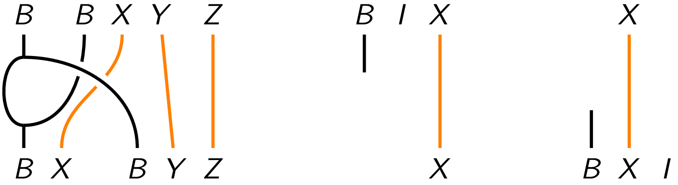


"Warped" tensor product $X * Y := B \otimes X \otimes Y$ with same unit



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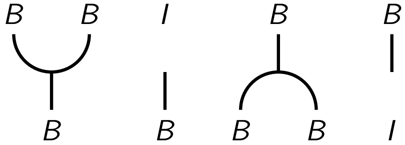

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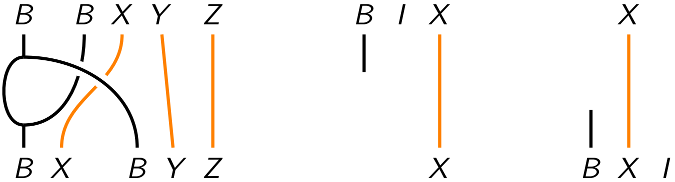

In **Vect**, can characterize bialgebras in terms of *closed* skew monoidal structures

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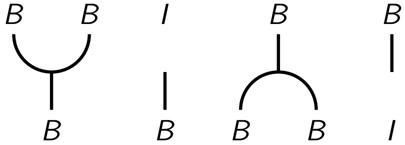

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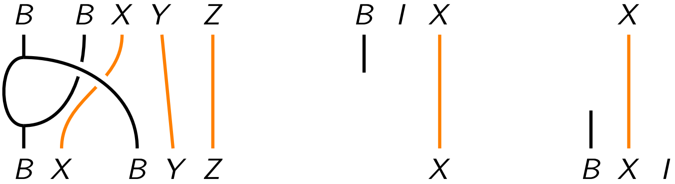

In **Vect**, can characterize bialgebras in terms of *closed* skew monoidal structures

And closed skew monoidal structures on **Mod**R correspond to *bialgebroids* with base algebra R.

2-categorical example

$\mathbf{FProd_s}$ is the 2-category consisting of

- categories with chosen finite products
- functors *strictly* preserving these
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$$\mathbf{FProd_s}(I \otimes A, B) \cong \mathbf{FProd_s}(I, [A, B]) \cong [A, B]$$

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$\mathbf{FProd_s}$ becomes skew monoidal (2-category)

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More generally, if T is an accessible pseudocommutative 2-monad on **Cat**, then there is a skew monoidal structure on the 2-category of T-algebras (with strict morphisms).
The unit is $T1$. Tensoring on the left with $T1$ classifies weak morphisms.

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- permutative categories
- braided monoidal categories categories equipped with an action by a fixed symmetric monoidal category
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Corollary

The 2-category of T-algebras with pseudo morphisms is a monoidal bicategory.

A symmetry for $\mathbf{FProd_s}$

$(A_1 \otimes A_2) \otimes A_3 \to B$ in $\mathbf{FProd_s}$ $\Leftrightarrow$ "trilinear" $A_1 \times A_2 \times A_3 \to B$
(strict in first variable)

Permuting 2nd and 3rd variables gives a new trilinear map

This induces isomorphisms

$$s\colon (A_1 \otimes A_2) \otimes A_3 \to (A_1 \otimes A_3) \otimes A_2$$

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More generally, if $A_1 A_2 \dots A_n$ is left-bracketed, have an action by all $\pi \in S_n$ which fix first element

Braided skew monoidal categories

A *braiding* on a skew monoidal category consists of natural isomorphisms

$$s\colon (XA)B \to (XB)A$$

subject to 4 coherence conditions including

$$\begin{array}{ccc}
((XA)B)C & \xrightarrow{a} & (XA)(BC) \\
{\scriptstyle s1}\downarrow & & \\
((XB)A)C & & \downarrow{\scriptstyle s} \\
{\scriptstyle s}\downarrow & & \\
((XB)C)A & \xrightarrow[a1]{} & (X(BC))A
\end{array}
\qquad
\begin{array}{ccc}
((XA)B)C & \xrightarrow{s} & ((XA)C)B \\
{\scriptstyle a1}\downarrow & & \downarrow{\scriptstyle a1} \\
(X(AB))C & & (X(AC))B \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\
X((AB)C) & \xrightarrow[1s]{} & X((AC)B)
\end{array}$$

(others are Yang-Baxter, and first of these for s^{-1})

If $s \circ s = 1$ then s is a *symmetry*.

Related structures

There are analogous notions of:

- skew closed category (Street)
- skew multicategory — involves *tight* and *loose* multimaps

$$(A_1 A_2)A_3 \xrightarrow{\text{“tight”}} B \qquad\qquad ((IA_1)A_2)A_3 \xrightarrow{\text{“loose”}} B$$

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Braidings make sense for these as well.

- $[X, [Y, Z]] \cong [Y, [X, Z]]$
- permuting inputs of multimaps

The fact that a braided skew monoidal category gives rise to a braided skew multicategory is a sort of coherence result.

Boring examples

Proposition

For an actual monoidal category, the two notions of braiding are equivalent.

Proof.

$$\begin{array}{ccc} (IA)B & \xrightarrow{s} & (IB)A \\ {\scriptstyle \ell 1}\downarrow & & \downarrow{\scriptstyle \ell 1} \\ AB & \xrightarrow[c]{} & BA \end{array}$$

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Proposition

A braided skew monoidal category for which the left unit map is invertible is monoidal.

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For bialgebra B in braided monoidal $\mathcal{V}$, recall that braidings on **Comod**B correspond to cobraidings (coquasitriangular structures) on B.

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Let $\mathcal{V}[B]$ be the warped skew monoidal structure with $X * Y = B \otimes X \otimes Y$.

Theorem

$\mathcal{V}[B]$ has a braiding if and only if B has a cobraiding.

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There are also results for more general skew warpings (not arising from a bialgebra).

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Braided monoidal bicategories

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Corollary

Our 2-categorical examples are symmetric monoidal bicategories.