Magnitude

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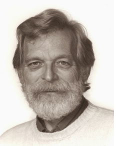

Stephen Schanuel:
Euler characteristic is the topological analogue of cardinality.

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One answer The magnitude of an enriched category.

1. The cardinality of a colimit

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Obviously not for an *arbitrary* X, but maybe under restrictions on X...

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E.g. A weighting on $Z_{\mathbf{A}}$ is a family $\big(w(a)\big)_{a \in \mathbf{A}}$ in $\mathbb{Q}$ such that

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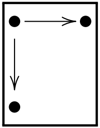
- $\mathbf{A} =$  :

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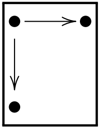
- $\mathbf{A} =$  : unique weighting is $\begin{array}{|cc|} -1 & 1 \\ & \\ 1 & \end{array}$

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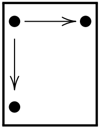
- $\mathbf{A} =$  : unique weighting is $\begin{array}{|cc|} -1 & 1 \\ & \\ 1 & \end{array}$, and Theorem gives the inclusion-exclusion formula.

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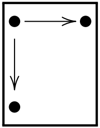
- $\mathbf{A} =$  : unique weighting is $\begin{array}{|cc|} -1 & 1 \\ & \\ 1 & \end{array}$, and Theorem gives the inclusion-exclusion formula.
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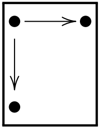
- $\mathbf{A} =$  : unique weighting is $\begin{array}{|cc|} -1 & 1 \\ & \\ 1 & \\ \end{array}$, and Theorem gives the inclusion-exclusion formula.
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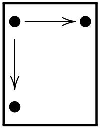
- $\mathbf{A} =$  : unique weighting is $\boxed{\begin{array}{cc} -1 & 1 \\ & \\ 1 & \end{array}}$, and Theorem gives the inclusion-exclusion formula.
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Ponto and Shulman have a more sophisticated version of the theorem.

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What does $\sum w(a)$ mean in general?

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Important special case If Z is invertible then it has a unique weighting, and

$$|Z| = \sum_{i,j} \left(Z^{-1}\right)_{ij}.$$

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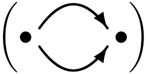
E.g. If $\mathbf{A} =$  then $B\mathbf{A} = S^1$ and $\chi(S^1) = 0 = |\mathbf{A}|$.

Other theorems connect magnitude of categories to Euler characteristic of manifolds — and more generally, orbifolds (whose Euler characteristics are usually $\notin \mathbb{Z}$).

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3. The magnitude of an enriched category

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Idea: Do the same with an arbitrary monoidal category in place of **FinSet**.

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E.g. Take $\mathscr{V} = \mathbf{FinSet}$, $k = \mathbb{Q}$, and $|\cdot| = \mathrm{card}$: then we recover the definition of the magnitude of a finite category.

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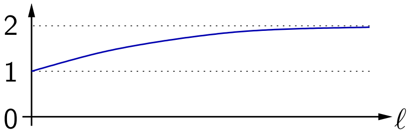

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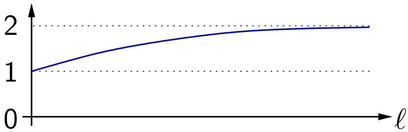

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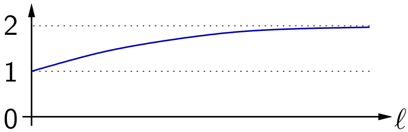

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Slogan: Magnitude is the 'effective number of points'

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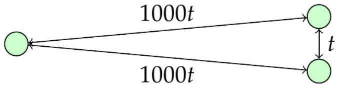
$A =$ 

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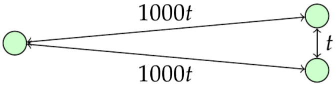
$A =$ 

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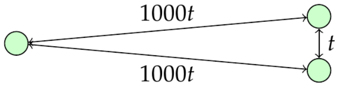
$A =$ 

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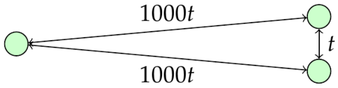
$A =$ 

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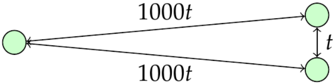
$A =$ 

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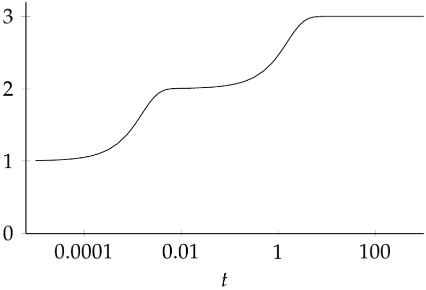
Indeed, the magnitude of A as a function of t is:



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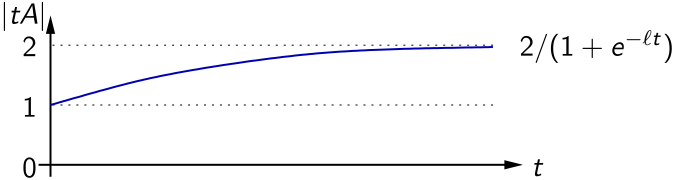

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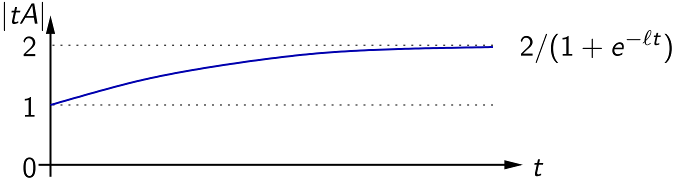

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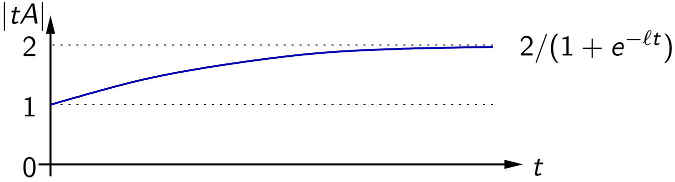

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It is increasing for $t \gg 0$, and $\lim_{t\to\infty} |tA| = \text{cardinality}(A)$.

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There's a similar theorem in higher dimensions.

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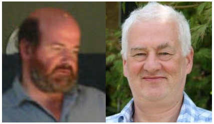

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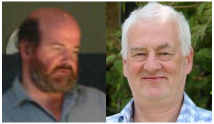

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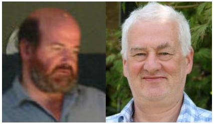

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Proof Uses heat trace asymptotics (techniques related to the heat equation proof of the Atiyah–Singer index theorem).

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But it *asymptotically* does.

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> Conclusion: Magnitude is closely related to maximum diversity.

End of digression

. . . back to magnitude of $\mathscr{V}$-categories

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It can also be expanded as a *power series* in x over $\mathbb{Z}$.

The magnitude of a graph: examples and theorems

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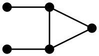
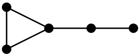
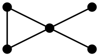
$\Big|$  $\Big| = \Big|$  $\Big| = \Big|$  $\Big| = \dfrac{5 + 5x - 4x^2}{(1 + x)(1 + 2x)}$

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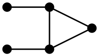
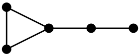
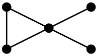
$\Bigg|$  $\Bigg| = \Bigg|$  $\Bigg| = \Bigg|$  $\Bigg| = \dfrac{5+5x-4x^2}{(1+x)(1+2x)}$

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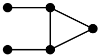
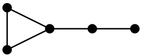
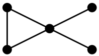
$\Bigg|$  $\Bigg| = \Bigg|$  $\Bigg| = \Bigg|$  $\Bigg| = \dfrac{5+5x-4x^2}{(1+x)(1+2x)}$

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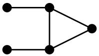
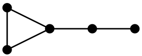
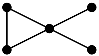
$\Big|$  $\Big| \;=\; \Big|$  $\Big| = \Big|$  $\Big| = \dfrac{5+5x-4x^2}{(1+x)(1+2x)}$

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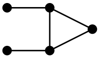
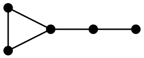
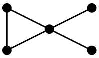
|  | = |  | = |  | $= \dfrac{5+5x-4x^2}{(1+x)(1+2x)}$

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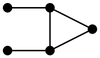
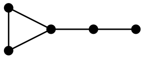
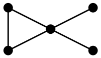
$\Bigg|$  $\Bigg| = \Bigg|$  $\Bigg| = \Bigg|$  $\Bigg| = \dfrac{5+5x-4x^2}{(1+x)(1+2x)}$

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- Graph magnitude has other invariance properties shared with the Tutte polynomial.

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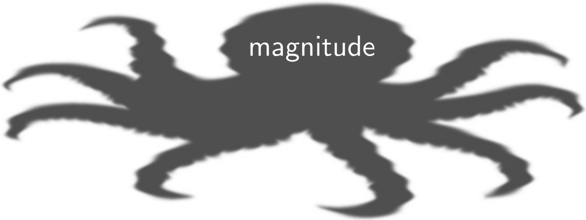
Also What about other bases $\mathscr{V}$ of enrichment?

4. Where's the category theory?

Overview

Overview



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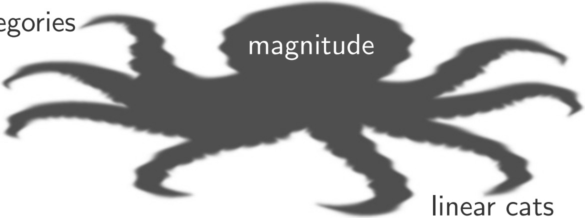
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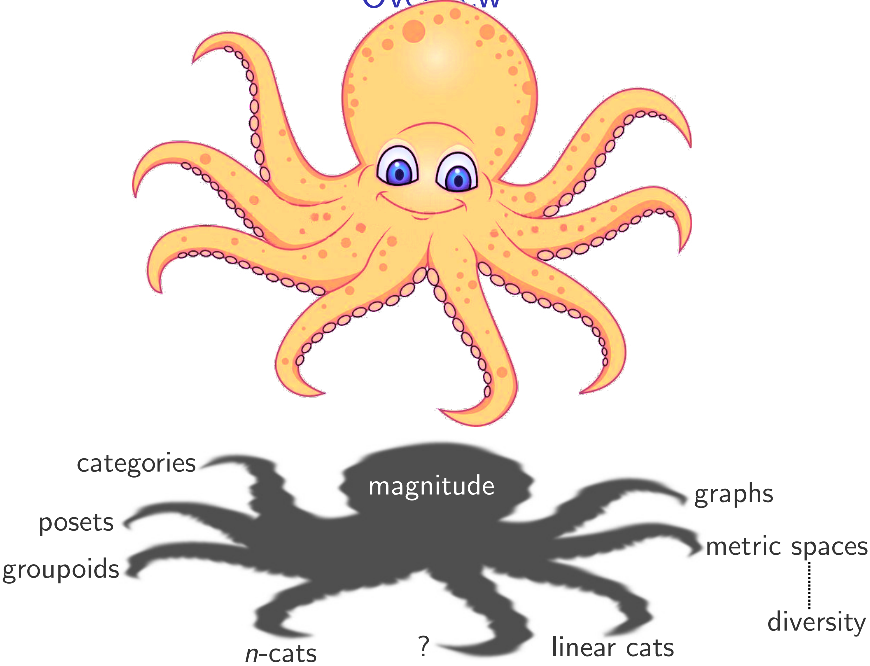
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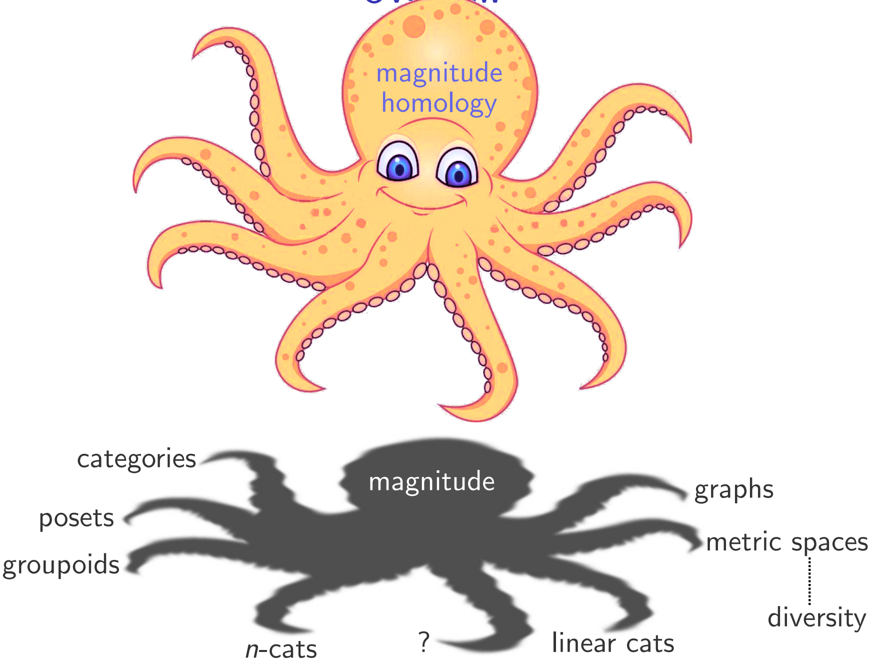
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5. Magnitude homology: a sketch

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In this sense, homology is a categorification of Euler characteristic.

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- Magnitude homology can distinguish between graphs that mere magnitude cannot.

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- There's an accompanying cohomology theory.

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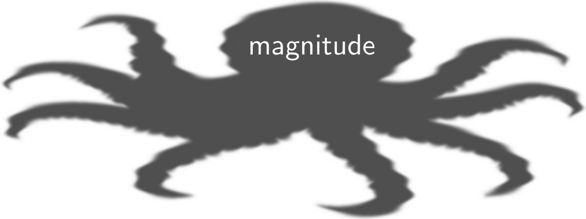
Very recent result of Nina Otter (arXiv paper last Wednesday):

magnitude homology is related to (but different from!) persistent homology.

Summary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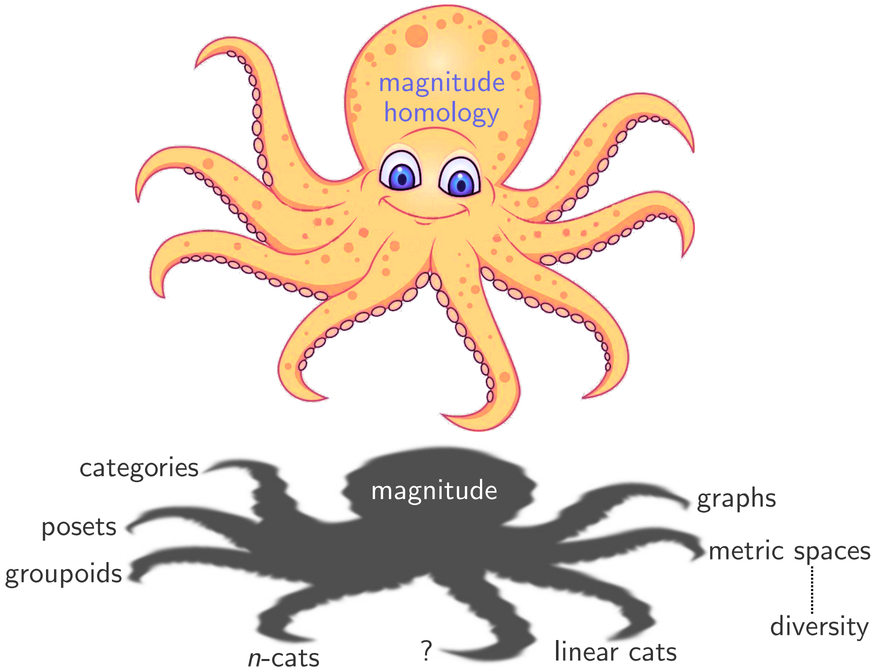
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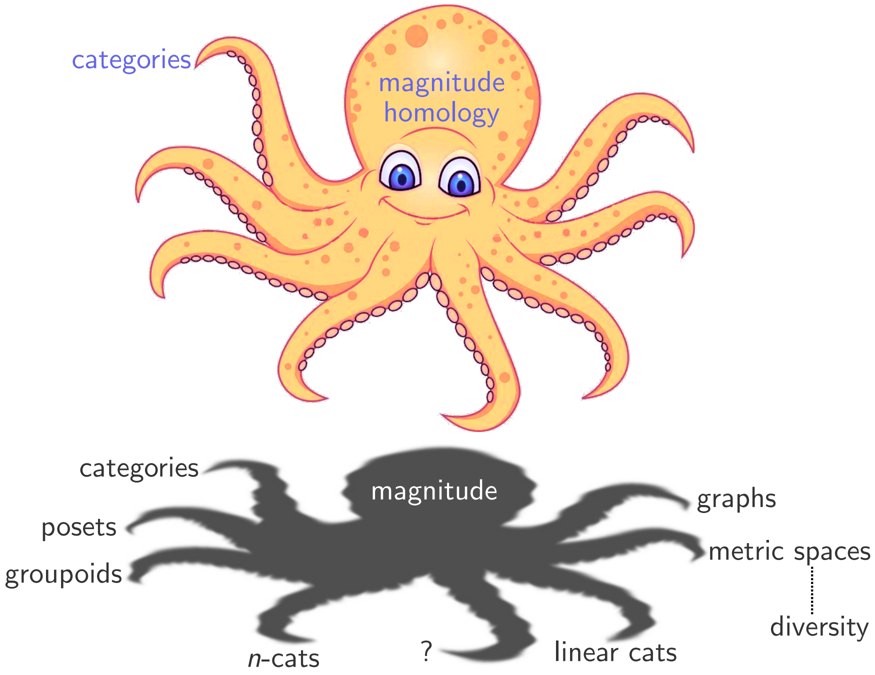
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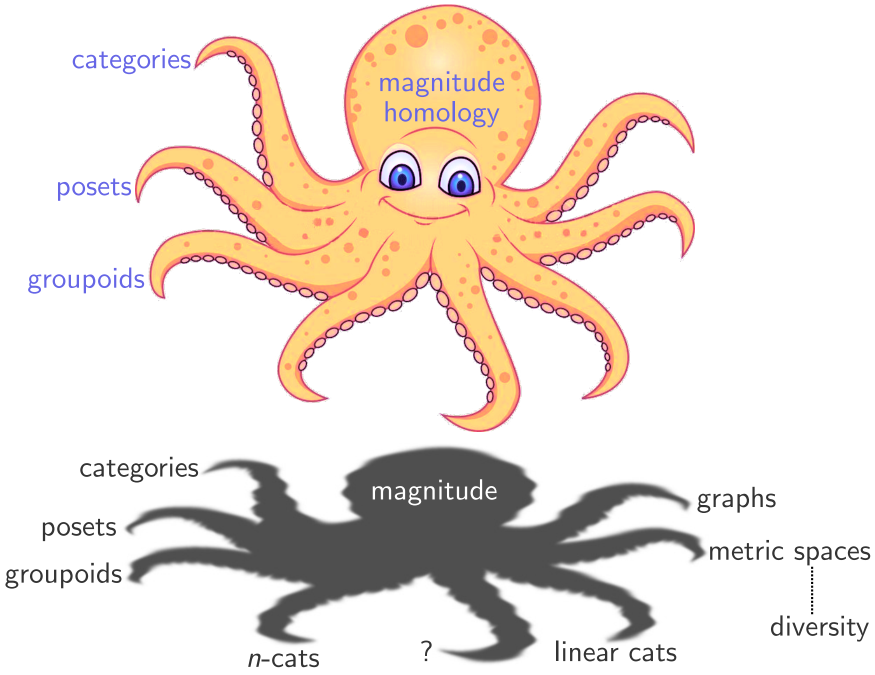
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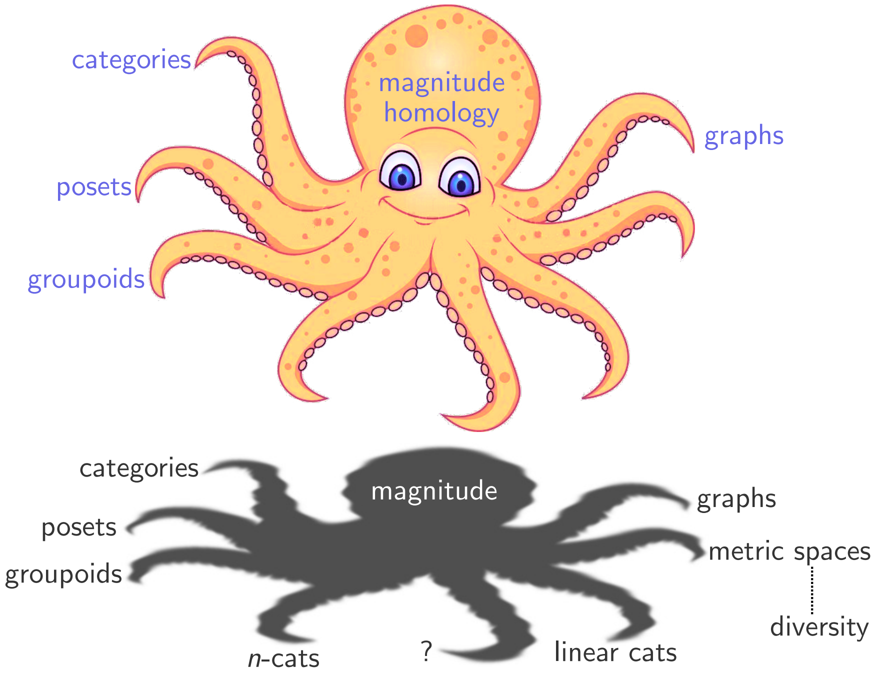
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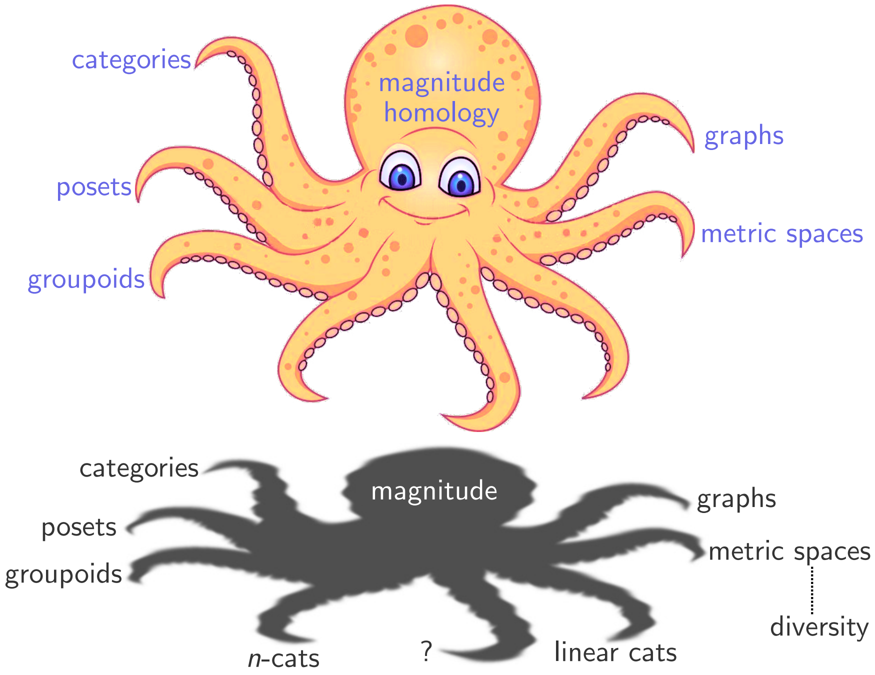
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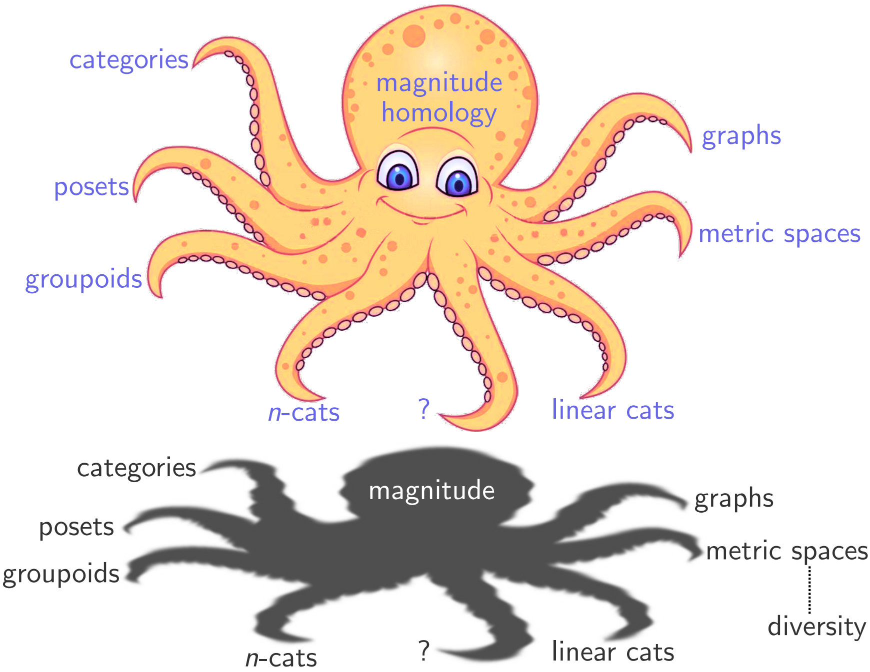
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Thanks

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 Jill Thompson

 Simon Willerton

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