USC
UNIVERSIDADE DE SANTIAGO DE COMPOSTELA
DEPARTAMENTO DE MATEMÁTICAS

A categorical characterisation of Lie algebras

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Joint work with Tim Van der Linden

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Ministerio de Economía, industria y Competitividad
MTM2016-79661-P Agencia Estatal de Investigación

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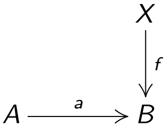

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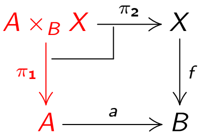

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Definition

$\mathcal{C}$ is locally cartesian closed if and only if all the change of base functors a^* have a right adjoint.

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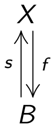

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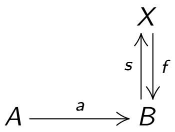

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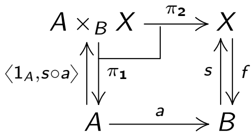

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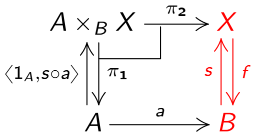

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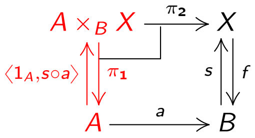

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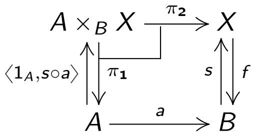

Definition (Gray, 2012)

$\mathcal{C}$ is locally algebraically cartesian closed (LACC for short) if and only if all the induced functors a^* have a right adjoint.

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Examples

- Abelian categories

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- Groups (over a cartesian closed category) (Gray, 2012)
- Lie algebras (over some monoidal categories) (Gray, 2012, G.M.-Gray, in progress)

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If $\mathcal{C}$ has zero object, then it is LACC if and only if

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has a right adjoint for all B.

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This defines a monad $B\flat(-)\colon \mathcal{C} \to \mathcal{C}$ that sends any X to

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An action of B on X (or a B-action) is an algebra over the monad $B\flat(-)$.
There is an equivalence of categories

$$\mathrm{Pt}_B(\mathcal{C}) \simeq B\text{-}\mathrm{Act}(\mathcal{C})$$

Non-associative algebras

Definition

Let $\mathbb{K}$ be a field. A non-associative algebra is a $\mathbb{K}$-vector space with a linear map

$$A \otimes A \to A.$$

We denote the category by $\mathtt{Alg}_{\mathbb{K}}$.

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A *subvariety* of $\mathtt{Alg}_{\mathbb{K}}$ is any equationally defined class of algebras, considered as a full subcategory $\mathcal{V}$ of $\mathtt{Alg}_{\mathbb{K}}$.

Non-associative algebras

Example

Lie algebras. They satisfy the equations

$$xx = 0$$

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Non-associative algebras

Theorem

If $\mathcal{V}$ is a variety of algebras over an infinite field $\mathbb{K}$, all of its identities are of the form $\phi(x_1, \ldots, x_n)$, where ϕ is a non-associative polynomial.
Moreover, each of its homogeneous components $\psi(x_{i_1}, \ldots, x_{i_n})$ is also an identity.

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Preservation of coproducts of $B\flat(-)$

Proposition (Gray, 2012)

Let $\mathcal{V}$ be a variety of non-associative algebras.
It is (LACC) *if and only if the canonical comparison*

$$(B\flat X + B\flat Y) \to B\flat(X + Y)$$

is an isomorphism.

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Theorem

The following are equivalent:

- $\mathcal{V}$ *is algebraically coherent,*
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- $\mathcal{V}$ *is an Orzech category of interest.*

Nilpotent algebras

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If $\mathcal{V}$ is (LACC) and $x(yz) = 0$ is an identity in $\mathcal{V}$, then $\mathcal{V}$ is abelian.

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Proof: Let B, X, Y be free algebras on one generator. Since $\mathcal{V}$ is (LACC), the morphism

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The element

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Then if $x(yb)$ is zero, either $x(yb) = 0$ or $yb = 0$ have to be rules of $\mathcal{V}$. In both cases, it implies that the algebra is abelian.

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Then $(xb)y$ and $x(by)$ go to the same element in $B\flat(X + Y)$ but they are different in $(B\flat X + B\flat Y)$.

Operations of degree 2

Theorem

If $\mathcal{V}$ is a (LACC) anticommutative variety of algebras, i.e. $xy = -yx$ is an identity, then $\mathcal{V}$ is subvariety of $\mathtt{Lie}_{\mathbb{K}}$.

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Theorem

Let $\mathbb{K}$ be an infinite field of char $\neq 2$.
If $\mathcal{V}$ is a $\mathrm{(LACC)}$ commutative variety of algebras, i.e. $xy = yx$ is an identity, then $\mathcal{V}$ is abelian.

Non-commutative and non-anticommutative

Let us assume that there are no operations of degree 2.

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is an isomorphism.

$$\begin{aligned} x(by) = {} & \lambda_1(xb)y + \lambda_2(bx)y + \lambda_3 y(xb) + \lambda_4 y(bx) \\ & + \lambda_5(xy)b + \lambda_6(yx)b + \lambda_7 b(xy) + \lambda_8 b(yx) \end{aligned}$$

$$\begin{aligned} = {} & \lambda_1(xb)y + \lambda_2(bx)y + \lambda_3 y(xb) + \lambda_4 y(bx) \\ & + \lambda_5\big(\mu_1(bx)y + \mu_2(xb)y + \mu_3 y(bx) + \cdots + \mu_7 x(by) + \mu_8 x(yb)\big) \\ & + \lambda_6\big(\mu_1(by)x + \mu_2(yb)x + \mu_3 x(by) + \cdots + \mu_7 y(bx) + \mu_8 y(xb)\big) \end{aligned}$$

Non-commutative and non-anticommutative

Let us assume that there are no operations of degree 2.
We need to see if there is any variety such that the map

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$$\begin{aligned}
x(by) &= \lambda_1(xb)y + \lambda_2(bx)y + \lambda_3 y(xb) + \lambda_4 y(bx) \\
&\quad + \lambda_5(xy)b + \lambda_6(yx)b + \lambda_7 b(xy) + \lambda_8 b(yx) \\
&= \lambda_1(xb)y + \lambda_2(bx)y + \lambda_3 y(xb) + \lambda_4 y(bx) \\
&\quad + \lambda_5\big(\mu_1(bx)y + \mu_2(xb)y + \mu_3 y(bx) + \cdots + \mu_7 x(by) + \mu_8 x(yb)\big) \\
&\quad + \lambda_6\big(\mu_1(by)x + \mu_2(yb)x + \mu_3 x(by) + \cdots + \mu_7 y(bx) + \mu_8 y(xb)\big) \\
&\quad + \lambda_7\big(\lambda_1(bx)y + \lambda_2(xb)y + \lambda_3 y(bx) + \cdots + \lambda_7 x(by) + \lambda_8 x(yb)\big)
\end{aligned}$$

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$$\begin{aligned} &= \lambda_1(xb)y + \lambda_2(bx)y + \lambda_3 y(xb) + \lambda_4 y(bx) \\ &+ \lambda_5\big(\mu_1(bx)y + \mu_2(xb)y + \mu_3 y(bx) + \cdots + \mu_7 x(by) + \mu_8 x(yb)\big) \\ &+ \lambda_6\big(\mu_1(by)x + \mu_2(yb)x + \mu_3 x(by) + \cdots + \mu_7 y(bx) + \mu_8 y(xb)\big) \\ &+ \lambda_7\big(\lambda_1(bx)y + \lambda_2(xb)y + \lambda_3 y(bx) + \cdots + \lambda_7 x(by) + \lambda_8 x(yb)\big) \end{aligned}$$

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$$= \lambda_1(xb)y + \lambda_2(bx)y + \lambda_3 y(xb) + \lambda_4 y(bx)$$
$$+ \lambda_5\big(\mu_1(bx)y + \mu_2(xb)y + \mu_3 y(bx) + \cdots + \mu_7 x(by) + \mu_8 x(yb)\big)$$
$$+ \lambda_6\big(\mu_1(by)x + \mu_2(yb)x + \mu_3 x(by) + \cdots + \mu_7 y(bx) + \mu_8 y(xb)\big)$$
$$+ \lambda_7\big(\lambda_1(bx)y + \lambda_2(xb)y + \lambda_3 y(bx) + \cdots + \lambda_7 x(by) + \lambda_8 x(yb)\big)$$
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$$\begin{aligned} = {} & \lambda_1(xb)y + \lambda_2(bx)y + \lambda_3 y(xb) + \lambda_4 y(bx) \\ & + \lambda_5\big(\mu_1(bx)y + \mu_2(xb)y + \mu_3 y(bx) + \cdots + \mu_7 x(by) + \mu_8 x(yb)\big) \\ & + \lambda_6\big(\mu_1(by)x + \mu_2(yb)x + \mu_3 x(by) + \cdots + \mu_7 y(bx) + \mu_8 y(xb)\big) \\ & + \lambda_7\big(\lambda_1(bx)y + \lambda_2(xb)y + \lambda_3 y(bx) + \cdots + \lambda_7 x(by) + \lambda_8 x(yb)\big) \\ & + \lambda_8\big(\lambda_1(by)x + \lambda_2(yb)x + \lambda_3 x(by) + \cdots + \lambda_7 y(bx) + \mu_8 y(xb)\big) \end{aligned}$$

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&\quad + \lambda_5(xy)b + \lambda_6(yx)b + \lambda_7 b(xy) + \lambda_8 b(yx) \\
&= \lambda_1(xb)y + \lambda_2(bx)y + \lambda_3 y(xb) + \lambda_4 y(bx) \\
&\quad + \lambda_5\big(\mu_1(bx)y + \mu_2(xb)y + \mu_3 y(bx) + \cdots + \mu_7 x(by) + \mu_8 x(yb)\big) \\
&\quad + \lambda_6\big(\mu_1(by)x + \mu_2(yb)x + \mu_3 x(by) + \cdots + \mu_7 y(bx) + \mu_8 y(xb)\big) \\
&\quad + \lambda_7\big(\lambda_1(bx)y + \lambda_2(xb)y + \lambda_3 y(bx) + \cdots + \lambda_7 x(by) + \lambda_8 x(yb)\big) \\
&\quad + \lambda_8\big(\lambda_1(by)x + \lambda_2(yb)x + \lambda_3 x(by) + \cdots + \lambda_7 y(bx) + \mu_8 y(xb)\big)
\end{aligned}$$

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$$\begin{aligned} &= \lambda_1(xb)y + \lambda_2(bx)y + \lambda_3 y(xb) + \lambda_4 y(bx) \\ &\quad + \lambda_5\big(\mu_1(bx)y + \mu_2(xb)y + \mu_3 y(bx) + \cdots + \mu_7 x(by) + \mu_8 x(yb)\big) \\ &\quad + \lambda_6\big(\mu_1(by)x + \mu_2(yb)x + \mu_3 x(by) + \cdots + \mu_7 y(bx) + \mu_8 y(xb)\big) \\ &\quad + \lambda_7\big(\lambda_1(bx)y + \lambda_2(xb)y + \lambda_3 y(bx) + \cdots + \lambda_7 x(by) + \lambda_8 x(yb)\big) \\ &\quad + \lambda_8\big(\lambda_1(by)x + \lambda_2(yb)x + \lambda_3 x(by) + \cdots + \lambda_7 y(bx) + \mu_8 y(xb)\big) \end{aligned}$$

$$
\begin{aligned}
f_{33} = {} & \mu_1\mu_1\mu_5 + \mu_2\lambda_1\mu_5 + \mu_3\mu_1\lambda_5 + \mu_4\lambda_1\lambda_5 + \mu_1\mu_3\mu_1 + \mu_2\lambda_3\mu_1 + \mu_3\mu_3\lambda_1 + \mu_4\lambda_3\lambda_1 \\
& + \mu_5\mu_5\mu_7 + \mu_6\lambda_5\mu_7 + \mu_7\mu_5\lambda_7 + \mu_8\lambda_5\lambda_7 + \mu_5\mu_7\mu_3 + \mu_6\lambda_7\mu_3 + \mu_7\mu_7\lambda_3 + \mu_8\lambda_7\lambda_3 \\
f_{34} = {} & \mu_1\mu_1\mu_6 + \mu_2\lambda_1\mu_6 + \mu_3\mu_1\lambda_6 + \mu_4\lambda_1\lambda_6 + \mu_1\mu_3\mu_2 + \mu_2\lambda_3\mu_2 + \mu_3\mu_3\lambda_2 + \mu_4\lambda_3\lambda_2 \\
& + \mu_5\mu_6\mu_7 + \mu_6\lambda_6\mu_7 + \mu_7\mu_6\lambda_7 + \mu_8\lambda_6\lambda_7 + \mu_5\mu_8\mu_3 + \mu_6\lambda_8\mu_3 + \mu_7\mu_8\lambda_3 + \mu_8\lambda_8\lambda_3 \\
f_{35} = {} & \mu_1\mu_1\mu_7 + \mu_2\lambda_1\mu_7 + \mu_3\mu_1\lambda_7 + \mu_4\lambda_1\lambda_7 + \mu_1\mu_3\mu_3 + \mu_2\lambda_3\mu_3 + \mu_3\mu_3\lambda_3 + \mu_4\lambda_3\lambda_3 \\
& + \mu_5\mu_5\mu_5 + \mu_6\lambda_5\mu_5 + \mu_7\mu_5\lambda_5 + \mu_8\lambda_5\lambda_5 + \mu_5\mu_7\mu_1 + \mu_6\lambda_7\mu_1 + \mu_7\mu_7\lambda_1 + \mu_8\lambda_7\lambda_1 \\
f_{36} = {} & \mu_1\mu_1\mu_8 + \mu_2\lambda_1\mu_8 + \mu_3\mu_1\lambda_8 + \mu_4\lambda_1\lambda_8 + \mu_1\mu_3\mu_4 + \mu_2\lambda_3\mu_4 + \mu_3\mu_3\lambda_4 + \mu_4\lambda_3\lambda_4 \\
& + \mu_5\mu_6\mu_5 + \mu_6\lambda_6\mu_5 + \mu_7\mu_6\lambda_5 + \mu_8\lambda_6\lambda_5 + \mu_5\mu_8\mu_1 + \mu_6\lambda_8\mu_1 + \mu_7\mu_8\lambda_1 + \mu_8\lambda_8\lambda_1 \\
f_{37} = {} & \mu_1\mu_2\mu_5 + \mu_2\lambda_2\mu_5 + \mu_3\mu_2\lambda_5 + \mu_4\lambda_2\lambda_5 + \mu_1\mu_4\mu_1 + \mu_2\lambda_4\mu_1 + \mu_3\mu_4\lambda_1 + \mu_4\lambda_4\lambda_1 \\
& + \mu_5\mu_5\mu_8 + \mu_6\lambda_5\mu_8 + \mu_7\mu_5\lambda_8 + \mu_8\lambda_5\lambda_8 + \mu_5\mu_7\mu_4 + \mu_6\lambda_7\mu_4 + \mu_7\mu_7\lambda_4 + \mu_8\lambda_7\lambda_4 \\
f_{38} = {} & \mu_1\mu_2\mu_6 + \mu_2\lambda_2\mu_6 + \mu_3\mu_2\lambda_6 + \mu_4\lambda_2\lambda_6 + \mu_1\mu_4\mu_2 + \mu_2\lambda_4\mu_2 + \mu_3\mu_4\lambda_2 + \mu_4\lambda_4\lambda_2 \\
& + \mu_5\mu_6\mu_8 + \mu_6\lambda_6\mu_8 + \mu_7\mu_6\lambda_8 + \mu_8\lambda_6\lambda_8 + \mu_5\mu_8\mu_4 + \mu_6\lambda_8\mu_4 + \mu_7\mu_8\lambda_4 + \mu_8\lambda_8\lambda_4 \\
f_{39} = {} & \mu_1\mu_2\mu_7 + \mu_2\lambda_2\mu_7 + \mu_3\mu_2\lambda_7 + \mu_4\lambda_2\lambda_7 + \mu_1\mu_4\mu_3 + \mu_2\lambda_4\mu_3 + \mu_3\mu_4\lambda_3 + \mu_4\lambda_4\lambda_3 \\
& + \mu_5\mu_5\mu_6 + \mu_6\lambda_5\mu_6 + \mu_7\mu_5\lambda_6 + \mu_8\lambda_5\lambda_6 + \mu_5\mu_7\mu_2 + \mu_6\lambda_7\mu_2 + \mu_7\mu_7\lambda_2 + \mu_8\lambda_7\lambda_2 \\
f_{40} = {} & \mu_1\mu_2\mu_8 + \mu_2\lambda_2\mu_8 + \mu_3\mu_2\lambda_8 + \mu_4\lambda_2\lambda_8 + \mu_1\mu_4\mu_4 + \mu_2\lambda_4\mu_4 + \mu_3\mu_4\lambda_4 + \mu_4\lambda_4\lambda_4 \\
& + \mu_5\mu_6\mu_6 + \mu_6\lambda_6\mu_6 + \mu_7\mu_6\lambda_6 + \mu_8\lambda_6\lambda_6 + \mu_5\mu_8\mu_2 + \mu_6\lambda_8\mu_2 + \mu_7\mu_8\lambda_2 + \mu_8\lambda_8\lambda_2
\end{aligned}
$$

$$
\begin{aligned}
f_{41} &= \mu_1\mu_5\mu_5 + \mu_2\lambda_5\mu_5 + \mu_3\mu_5\lambda_5 + \mu_4\lambda_5\lambda_5 + \mu_1\mu_7\mu_1 + \mu_2\lambda_7\mu_1 + \mu_3\mu_7\lambda_1 + \mu_4\lambda_7\lambda_1 \\
&\quad + \mu_5\mu_1\mu_7 + \mu_6\lambda_1\mu_7 + \mu_7\mu_1\lambda_7 + \mu_8\lambda_1\lambda_7 + \mu_5\mu_3\mu_3 + \mu_6\lambda_3\mu_3 + \mu_7\mu_3\lambda_3 + \mu_8\lambda_3\lambda_3 \\
f_{42} &= \mu_1\mu_5\mu_6 + \mu_2\lambda_5\mu_6 + \mu_3\mu_5\lambda_6 + \mu_4\lambda_5\lambda_6 + \mu_1\mu_7\mu_2 + \mu_2\lambda_7\mu_2 + \mu_3\mu_7\lambda_2 + \mu_4\lambda_7\lambda_2 \\
&\quad + \mu_5\mu_2\mu_7 + \mu_6\lambda_2\mu_7 + \mu_7\mu_2\lambda_7 + \mu_8\lambda_2\lambda_7 + \mu_5\mu_4\mu_3 + \mu_6\lambda_4\mu_3 + \mu_7\mu_4\lambda_3 + \mu_8\lambda_4\lambda_3 \\
f_{43} &= \mu_1\mu_5\mu_7 + \mu_2\lambda_5\mu_7 + \mu_3\mu_5\lambda_7 + \mu_4\lambda_5\lambda_7 + \mu_1\mu_7\mu_3 + \mu_2\lambda_7\mu_3 + \mu_3\mu_7\lambda_3 + \mu_4\lambda_7\lambda_3 \\
&\quad + \mu_5\mu_1\mu_5 + \mu_6\lambda_1\mu_5 + \mu_7\mu_1\lambda_5 + \mu_8\lambda_1\lambda_5 + \mu_5\mu_3\mu_1 + \mu_6\lambda_3\mu_1 + \mu_7\mu_3\lambda_1 + \mu_8\lambda_3\lambda_1 \\
f_{44} &= \mu_1\mu_5\mu_8 + \mu_2\lambda_5\mu_8 + \mu_3\mu_5\lambda_8 + \mu_4\lambda_5\lambda_8 + \mu_1\mu_7\mu_4 + \mu_2\lambda_7\mu_4 + \mu_3\mu_7\lambda_4 + \mu_4\lambda_7\lambda_4 \\
&\quad + \mu_5\mu_2\mu_5 + \mu_6\lambda_2\mu_5 + \mu_7\mu_2\lambda_5 + \mu_8\lambda_2\lambda_5 + \mu_5\mu_4\mu_1 + \mu_6\lambda_4\mu_1 + \mu_7\mu_4\lambda_1 + \mu_8\lambda_4\lambda_1 \\
f_{45} &= \mu_1\mu_6\mu_5 + \mu_2\lambda_6\mu_5 + \mu_3\mu_6\lambda_5 + \mu_4\lambda_6\lambda_5 + \mu_1\mu_8\mu_1 + \mu_2\lambda_8\mu_1 + \mu_3\mu_8\lambda_1 + \mu_4\lambda_8\lambda_1 \\
&\quad + \mu_5\mu_1\mu_8 + \mu_6\lambda_1\mu_8 + \mu_7\mu_1\lambda_8 + \mu_8\lambda_1\lambda_8 + \mu_5\mu_3\mu_4 + \mu_6\lambda_3\mu_4 + \mu_7\mu_3\lambda_4 + \mu_8\lambda_3\lambda_4 \\
f_{46} &= \mu_1\mu_6\mu_6 + \mu_2\lambda_6\mu_6 + \mu_3\mu_6\lambda_6 + \mu_4\lambda_6\lambda_6 + \mu_1\mu_8\mu_2 + \mu_2\lambda_8\mu_2 + \mu_3\mu_8\lambda_2 + \mu_4\lambda_8\lambda_2 \\
&\quad + \mu_5\mu_2\mu_8 + \mu_6\lambda_2\mu_8 + \mu_7\mu_2\lambda_8 + \mu_8\lambda_2\lambda_8 + \mu_5\mu_4\mu_4 + \mu_6\lambda_4\mu_4 + \mu_7\mu_4\lambda_4 + \mu_8\lambda_4\lambda_4 \\
f_{47} &= \mu_1\mu_6\mu_7 + \mu_2\lambda_6\mu_7 + \mu_3\mu_6\lambda_7 + \mu_4\lambda_6\lambda_7 + \mu_1\mu_8\mu_3 + \mu_2\lambda_8\mu_3 + \mu_3\mu_8\lambda_3 + \mu_4\lambda_8\lambda_3 \\
&\quad + \mu_5\mu_1\mu_6 + \mu_6\lambda_1\mu_6 + \mu_7\mu_1\lambda_6 + \mu_8\lambda_1\lambda_6 + \mu_5\mu_3\mu_2 + \mu_6\lambda_3\mu_2 + \mu_7\mu_3\lambda_2 + \mu_8\lambda_3\lambda_2 \\
f_{48} &= \mu_1\mu_6\mu_8 + \mu_2\lambda_6\mu_8 + \mu_3\mu_6\lambda_8 + \mu_4\lambda_6\lambda_8 + \mu_1\mu_8\mu_4 + \mu_2\lambda_8\mu_4 + \mu_3\mu_8\lambda_4 + \mu_4\lambda_8\lambda_4 \\
&\quad + \mu_5\mu_2\mu_6 + \mu_6\lambda_2\mu_6 + \mu_7\mu_2\lambda_6 + \mu_8\lambda_2\lambda_6 + \mu_5\mu_4\mu_2 + \mu_6\lambda_4\mu_2 + \mu_7\mu_4\lambda_2 + \mu_8\lambda_4\lambda_2
\end{aligned}
$$

$$f_{49} = -\lambda_1\mu_1 - \lambda_2\lambda_1 + \mu_1\mu_2\mu_2 + \mu_2\lambda_2\mu_2 + \mu_3\mu_2\lambda_2 + \mu_4\lambda_2\lambda_2 + \mu_1\mu_4\mu_6 + \mu_2\lambda_4\mu_6 + \mu_3\mu_4\lambda_6 + \mu_4\lambda_4\lambda_6$$
$$f_{50} = -\lambda_1\mu_2 - \lambda_2\lambda_2 + \mu_1\mu_1\mu_2 + \mu_2\lambda_1\mu_2 + \mu_3\mu_1\lambda_2 + \mu_4\lambda_1\lambda_2 + \mu_1\mu_3\mu_6 + \mu_2\lambda_3\mu_6 + \mu_3\mu_3\lambda_6 + \mu_4\lambda_3\lambda_6$$
$$f_{51} = -\lambda_1\mu_3 - \lambda_2\lambda_3 + \mu_1\mu_2\mu_1 + \mu_2\lambda_2\mu_1 + \mu_3\mu_2\lambda_1 + \mu_4\lambda_2\lambda_1 + \mu_1\mu_4\mu_5 + \mu_2\lambda_4\mu_5 + \mu_3\mu_4\lambda_5 + \mu_4\lambda_4\lambda_5$$
$$f_{52} = -\lambda_1\mu_4 - \lambda_2\lambda_4 + \mu_1\mu_1\mu_1 + \mu_2\lambda_1\mu_1 + \mu_3\mu_1\lambda_1 + \mu_4\lambda_1\lambda_1 + \mu_1\mu_3\mu_5 + \mu_2\lambda_3\mu_5 + \mu_3\mu_3\lambda_5 + \mu_4\lambda_3\lambda_5$$
$$f_{53} = -\lambda_1\mu_5 - \lambda_2\lambda_5 + \mu_5\mu_2\mu_2 + \mu_6\lambda_2\mu_2 + \mu_7\mu_2\lambda_2 + \mu_8\lambda_2\lambda_2 + \mu_5\mu_4\mu_6 + \mu_6\lambda_4\mu_6 + \mu_7\mu_4\lambda_6 + \mu_8\lambda_4\lambda_6$$
$$f_{54} = -\lambda_1\mu_6 - \lambda_2\lambda_6 + \mu_5\mu_1\mu_2 + \mu_6\lambda_1\mu_2 + \mu_7\mu_1\lambda_2 + \mu_8\lambda_1\lambda_2 + \mu_5\mu_3\mu_6 + \mu_6\lambda_3\mu_6 + \mu_7\mu_3\lambda_6 + \mu_8\lambda_3\lambda_6$$
$$f_{55} = -\lambda_1\mu_7 - \lambda_2\lambda_7 + \mu_5\mu_2\mu_1 + \mu_6\lambda_2\mu_1 + \mu_7\mu_2\lambda_1 + \mu_8\lambda_2\lambda_1 + \mu_5\mu_4\mu_5 + \mu_6\lambda_4\mu_5 + \mu_7\mu_4\lambda_5 + \mu_8\lambda_4\lambda_5$$
$$f_{56} = -\lambda_1\mu_8 - \lambda_2\lambda_8 + \mu_5\mu_1\mu_1 + \mu_6\lambda_1\mu_1 + \mu_7\mu_1\lambda_1 + \mu_8\lambda_1\lambda_1 + \mu_5\mu_3\mu_5 + \mu_6\lambda_3\mu_5 + \mu_7\mu_3\lambda_5 + \mu_8\lambda_3\lambda_5$$
$$f_{57} = -\lambda_3\mu_1 - \lambda_4\lambda_1 + \mu_1\mu_2\mu_4 + \mu_2\lambda_2\mu_4 + \mu_3\mu_2\lambda_4 + \mu_4\lambda_2\lambda_4 + \mu_1\mu_4\mu_8 + \mu_2\lambda_4\mu_8 + \mu_3\mu_4\lambda_8 + \mu_4\lambda_4\lambda_8$$
$$f_{58} = -\lambda_3\mu_2 - \lambda_4\lambda_2 + \mu_1\mu_1\mu_4 + \mu_2\lambda_1\mu_4 + \mu_3\mu_1\lambda_4 + \mu_4\lambda_1\lambda_4 + \mu_1\mu_3\mu_8 + \mu_2\lambda_3\mu_8 + \mu_3\mu_3\lambda_8 + \mu_4\lambda_3\lambda_8$$
$$f_{59} = -\lambda_3\mu_3 - \lambda_4\lambda_3 + \mu_1\mu_2\mu_3 + \mu_2\lambda_2\mu_3 + \mu_3\mu_2\lambda_3 + \mu_4\lambda_2\lambda_3 + \mu_1\mu_4\mu_7 + \mu_2\lambda_4\mu_7 + \mu_3\mu_4\lambda_7 + \mu_4\lambda_4\lambda_7$$
$$f_{60} = -\lambda_3\mu_4 - \lambda_4\lambda_4 + \mu_1\mu_1\mu_3 + \mu_2\lambda_1\mu_3 + \mu_3\mu_1\lambda_3 + \mu_4\lambda_1\lambda_3 + \mu_1\mu_3\mu_7 + \mu_2\lambda_3\mu_7 + \mu_3\mu_3\lambda_7 + \mu_4\lambda_3\lambda_7$$
$$f_{61} = -\lambda_3\mu_5 - \lambda_4\lambda_5 + \mu_5\mu_2\mu_4 + \mu_6\lambda_2\mu_4 + \mu_7\mu_2\lambda_4 + \mu_8\lambda_2\lambda_4 + \mu_5\mu_4\mu_8 + \mu_6\lambda_4\mu_8 + \mu_7\mu_4\lambda_8 + \mu_8\lambda_4\lambda_8$$
$$f_{62} = -\lambda_3\mu_6 - \lambda_4\lambda_6 + \mu_5\mu_1\mu_4 + \mu_6\lambda_1\mu_4 + \mu_7\mu_1\lambda_4 + \mu_8\lambda_1\lambda_4 + \mu_5\mu_3\mu_8 + \mu_6\lambda_3\mu_8 + \mu_7\mu_3\lambda_8 + \mu_8\lambda_3\lambda_8$$
$$f_{63} = -\lambda_3\mu_7 - \lambda_4\lambda_7 + \mu_5\mu_2\mu_3 + \mu_6\lambda_2\mu_3 + \mu_7\mu_2\lambda_3 + \mu_8\lambda_2\lambda_3 + \mu_5\mu_4\mu_7 + \mu_6\lambda_4\mu_7 + \mu_7\mu_4\lambda_7 + \mu_8\lambda_4\lambda_7$$
$$f_{64} = -\lambda_3\mu_8 - \lambda_4\lambda_8 + \mu_5\mu_1\mu_3 + \mu_6\lambda_1\mu_3 + \mu_7\mu_1\lambda_3 + \mu_8\lambda_1\lambda_3 + \mu_5\mu_3\mu_7 + \mu_6\lambda_3\mu_7 + \mu_7\mu_3\lambda_7 + \mu_8\lambda_3\lambda_7$$

$$
\begin{aligned}
f_{65} &= -\lambda_5\mu_1 - \lambda_6\lambda_1 + \mu_1\mu_6\mu_2 + \mu_2\lambda_6\mu_2 + \mu_3\mu_6\lambda_2 + \mu_4\lambda_6\lambda_2 + \mu_1\mu_8\mu_6 + \mu_2\lambda_8\mu_6 + \mu_3\mu_8\lambda_6 + \mu_4\lambda_8\lambda_6 \\
f_{66} &= -\lambda_5\mu_2 - \lambda_6\lambda_2 + \mu_1\mu_5\mu_2 + \mu_2\lambda_5\mu_2 + \mu_3\mu_5\lambda_2 + \mu_4\lambda_5\lambda_2 + \mu_1\mu_7\mu_6 + \mu_2\lambda_7\mu_6 + \mu_3\mu_7\lambda_6 + \mu_4\lambda_7\lambda_6 \\
f_{67} &= -\lambda_5\mu_3 - \lambda_6\lambda_3 + \mu_1\mu_6\mu_1 + \mu_2\lambda_6\mu_1 + \mu_3\mu_6\lambda_1 + \mu_4\lambda_6\lambda_1 + \mu_1\mu_8\mu_5 + \mu_2\lambda_8\mu_5 + \mu_3\mu_8\lambda_5 + \mu_4\lambda_8\lambda_5 \\
f_{68} &= -\lambda_5\mu_4 - \lambda_6\lambda_4 + \mu_1\mu_5\mu_1 + \mu_2\lambda_5\mu_1 + \mu_3\mu_5\lambda_1 + \mu_4\lambda_5\lambda_1 + \mu_1\mu_7\mu_5 + \mu_2\lambda_7\mu_5 + \mu_3\mu_7\lambda_5 + \mu_4\lambda_7\lambda_5 \\
f_{69} &= -\lambda_5\mu_5 - \lambda_6\lambda_5 + \mu_5\mu_6\mu_2 + \mu_6\lambda_6\mu_2 + \mu_7\mu_6\lambda_2 + \mu_8\lambda_6\lambda_2 + \mu_5\mu_8\mu_6 + \mu_6\lambda_8\mu_6 + \mu_7\mu_8\lambda_6 + \mu_8\lambda_8\lambda_6 \\
f_{70} &= -\lambda_5\mu_6 - \lambda_6\lambda_6 + \mu_5\mu_5\mu_2 + \mu_6\lambda_5\mu_2 + \mu_7\mu_5\lambda_2 + \mu_8\lambda_5\lambda_2 + \mu_5\mu_7\mu_6 + \mu_6\lambda_7\mu_6 + \mu_7\mu_7\lambda_6 + \mu_8\lambda_7\lambda_6 \\
f_{71} &= -\lambda_5\mu_7 - \lambda_6\lambda_7 + \mu_5\mu_6\mu_1 + \mu_6\lambda_6\mu_1 + \mu_7\mu_6\lambda_1 + \mu_8\lambda_6\lambda_1 + \mu_5\mu_8\mu_5 + \mu_6\lambda_8\mu_5 + \mu_7\mu_8\lambda_5 + \mu_8\lambda_8\lambda_5 \\
f_{72} &= -\lambda_5\mu_8 - \lambda_6\lambda_8 + \mu_5\mu_5\mu_1 + \mu_6\lambda_5\mu_1 + \mu_7\mu_5\lambda_1 + \mu_8\lambda_5\lambda_1 + \mu_5\mu_7\mu_5 + \mu_6\lambda_7\mu_5 + \mu_7\mu_7\lambda_5 + \mu_8\lambda_7\lambda_5 \\
f_{73} &= -\lambda_7\mu_1 - \lambda_8\lambda_1 + \mu_1\mu_6\mu_4 + \mu_2\lambda_6\mu_4 + \mu_3\mu_6\lambda_4 + \mu_4\lambda_6\lambda_4 + \mu_1\mu_8\mu_8 + \mu_2\lambda_8\mu_8 + \mu_3\mu_8\lambda_8 + \mu_4\lambda_8\lambda_8 \\
f_{74} &= -\lambda_7\mu_2 - \lambda_8\lambda_2 + \mu_1\mu_5\mu_4 + \mu_2\lambda_5\mu_4 + \mu_3\mu_5\lambda_4 + \mu_4\lambda_5\lambda_4 + \mu_1\mu_7\mu_8 + \mu_2\lambda_7\mu_8 + \mu_3\mu_7\lambda_8 + \mu_4\lambda_7\lambda_8 \\
f_{75} &= -\lambda_7\mu_3 - \lambda_8\lambda_3 + \mu_1\mu_6\mu_3 + \mu_2\lambda_6\mu_3 + \mu_3\mu_6\lambda_3 + \mu_4\lambda_6\lambda_3 + \mu_1\mu_8\mu_7 + \mu_2\lambda_8\mu_7 + \mu_3\mu_8\lambda_7 + \mu_4\lambda_8\lambda_7 \\
f_{76} &= -\lambda_7\mu_4 - \lambda_8\lambda_4 + \mu_1\mu_5\mu_3 + \mu_2\lambda_5\mu_3 + \mu_3\mu_5\lambda_3 + \mu_4\lambda_5\lambda_3 + \mu_1\mu_7\mu_7 + \mu_2\lambda_7\mu_7 + \mu_3\mu_7\lambda_7 + \mu_4\lambda_7\lambda_7 \\
f_{77} &= -\lambda_7\mu_5 - \lambda_8\lambda_5 + \mu_5\mu_6\mu_4 + \mu_6\lambda_6\mu_4 + \mu_7\mu_6\lambda_4 + \mu_8\lambda_6\lambda_4 + \mu_5\mu_8\mu_8 + \mu_6\lambda_8\mu_8 + \mu_7\mu_8\lambda_8 + \mu_8\lambda_8\lambda_8 \\
f_{78} &= -\lambda_7\mu_6 - \lambda_8\lambda_6 + \mu_5\mu_5\mu_4 + \mu_6\lambda_5\mu_4 + \mu_7\mu_5\lambda_4 + \mu_8\lambda_5\lambda_4 + \mu_5\mu_7\mu_8 + \mu_6\lambda_7\mu_8 + \mu_7\mu_7\lambda_8 + \mu_8\lambda_7\lambda_8 \\
f_{79} &= -\lambda_7\mu_7 - \lambda_8\lambda_7 + \mu_5\mu_6\mu_3 + \mu_6\lambda_6\mu_3 + \mu_7\mu_6\lambda_3 + \mu_8\lambda_6\lambda_3 + \mu_5\mu_8\mu_7 + \mu_6\lambda_8\mu_7 + \mu_7\mu_8\lambda_7 + \mu_8\lambda_8\lambda_7 \\
f_{80} &= -\lambda_7\mu_8 - \lambda_8\lambda_8 + \mu_5\mu_5\mu_3 + \mu_6\lambda_5\mu_3 + \mu_7\mu_5\lambda_3 + \mu_8\lambda_5\lambda_3 + \mu_5\mu_7\mu_7 + \mu_6\lambda_7\mu_7 + \mu_7\mu_7\lambda_7 + \mu_8\lambda_7\lambda_7
\end{aligned}
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\begin{aligned}
f_{81} = {} & \lambda_1\mu_1\mu_5 + \lambda_2\lambda_1\mu_5 + \lambda_3\mu_1\lambda_5 + \lambda_4\lambda_1\lambda_5 + \lambda_1\mu_3\mu_1 + \lambda_2\lambda_3\mu_1 + \lambda_3\mu_3\lambda_1 + \lambda_4\lambda_3\lambda_1 \\
& + \lambda_5\mu_5\mu_7 + \lambda_6\lambda_5\mu_7 + \lambda_7\mu_5\lambda_7 + \lambda_8\lambda_5\lambda_7 + \lambda_5\mu_7\mu_3 + \lambda_6\lambda_7\mu_3 + \lambda_7\mu_7\lambda_3 + \lambda_8\lambda_7\lambda_3 \\
f_{82} = {} & \lambda_1\mu_1\mu_6 + \lambda_2\lambda_1\mu_6 + \lambda_3\mu_1\lambda_6 + \lambda_4\lambda_1\lambda_6 + \lambda_1\mu_3\mu_2 + \lambda_2\lambda_3\mu_2 + \lambda_3\mu_3\lambda_2 + \lambda_4\lambda_3\lambda_2 \\
& + \lambda_5\mu_6\mu_7 + \lambda_6\lambda_6\mu_7 + \lambda_7\mu_6\lambda_7 + \lambda_8\lambda_6\lambda_7 + \lambda_5\mu_8\mu_3 + \lambda_6\lambda_8\mu_3 + \lambda_7\mu_8\lambda_3 + \lambda_8\lambda_8\lambda_3 \\
f_{83} = {} & \lambda_1\mu_1\mu_7 + \lambda_2\lambda_1\mu_7 + \lambda_3\mu_1\lambda_7 + \lambda_4\lambda_1\lambda_7 + \lambda_1\mu_3\mu_3 + \lambda_2\lambda_3\mu_3 + \lambda_3\mu_3\lambda_3 + \lambda_4\lambda_3\lambda_3 \\
& + \lambda_5\mu_5\mu_5 + \lambda_6\lambda_5\mu_5 + \lambda_7\mu_5\lambda_5 + \lambda_8\lambda_5\lambda_5 + \lambda_5\mu_7\mu_1 + \lambda_6\lambda_7\mu_1 + \lambda_7\mu_7\lambda_1 + \lambda_8\lambda_7\lambda_1 \\
f_{84} = {} & \lambda_1\mu_1\mu_8 + \lambda_2\lambda_1\mu_8 + \lambda_3\mu_1\lambda_8 + \lambda_4\lambda_1\lambda_8 + \lambda_1\mu_3\mu_4 + \lambda_2\lambda_3\mu_4 + \lambda_3\mu_3\lambda_4 + \lambda_4\lambda_3\lambda_4 \\
& + \lambda_5\mu_6\mu_5 + \lambda_6\lambda_6\mu_5 + \lambda_7\mu_6\lambda_5 + \lambda_8\lambda_6\lambda_5 + \lambda_5\mu_8\mu_1 + \lambda_6\lambda_8\mu_1 + \lambda_7\mu_8\lambda_1 + \lambda_8\lambda_8\lambda_1 \\
f_{85} = {} & \lambda_1\mu_2\mu_5 + \lambda_2\lambda_2\mu_5 + \lambda_3\mu_2\lambda_5 + \lambda_4\lambda_2\lambda_5 + \lambda_1\mu_4\mu_1 + \lambda_2\lambda_4\mu_1 + \lambda_3\mu_4\lambda_1 + \lambda_4\lambda_4\lambda_1 \\
& + \lambda_5\mu_5\mu_8 + \lambda_6\lambda_5\mu_8 + \lambda_7\mu_5\lambda_8 + \lambda_8\lambda_5\lambda_8 + \lambda_5\mu_7\mu_4 + \lambda_6\lambda_7\mu_4 + \lambda_7\mu_7\lambda_4 + \lambda_8\lambda_7\lambda_4 \\
f_{86} = {} & \lambda_1\mu_2\mu_6 + \lambda_2\lambda_2\mu_6 + \lambda_3\mu_2\lambda_6 + \lambda_4\lambda_2\lambda_6 + \lambda_1\mu_4\mu_2 + \lambda_2\lambda_4\mu_2 + \lambda_3\mu_4\lambda_2 + \lambda_4\lambda_4\lambda_2 \\
& + \lambda_5\mu_6\mu_8 + \lambda_6\lambda_6\mu_8 + \lambda_7\mu_6\lambda_8 + \lambda_8\lambda_6\lambda_8 + \lambda_5\mu_8\mu_4 + \lambda_6\lambda_8\mu_4 + \lambda_7\mu_8\lambda_4 + \lambda_8\lambda_8\lambda_4 \\
f_{87} = {} & \lambda_1\mu_2\mu_7 + \lambda_2\lambda_2\mu_7 + \lambda_3\mu_2\lambda_7 + \lambda_4\lambda_2\lambda_7 + \lambda_1\mu_4\mu_3 + \lambda_2\lambda_4\mu_3 + \lambda_3\mu_4\lambda_3 + \lambda_4\lambda_4\lambda_3 \\
& + \lambda_5\mu_5\mu_6 + \lambda_6\lambda_5\mu_6 + \lambda_7\mu_5\lambda_6 + \lambda_8\lambda_5\lambda_6 + \lambda_5\mu_7\mu_2 + \lambda_6\lambda_7\mu_2 + \lambda_7\mu_7\lambda_2 + \lambda_8\lambda_7\lambda_2 \\
f_{88} = {} & \lambda_1\mu_2\mu_8 + \lambda_2\lambda_2\mu_8 + \lambda_3\mu_2\lambda_8 + \lambda_4\lambda_2\lambda_8 + \lambda_1\mu_4\mu_4 + \lambda_2\lambda_4\mu_4 + \lambda_3\mu_4\lambda_4 + \lambda_4\lambda_4\lambda_4 \\
& + \lambda_5\mu_6\mu_6 + \lambda_6\lambda_6\mu_6 + \lambda_7\mu_6\lambda_6 + \lambda_8\lambda_6\lambda_6 + \lambda_5\mu_8\mu_2 + \lambda_6\lambda_8\mu_2 + \lambda_7\mu_8\lambda_2 + \lambda_8\lambda_8\lambda_2
\end{aligned}
$$

$$
\begin{aligned}
f_{89} = {} & \lambda_1\mu_5\mu_5 + \lambda_2\lambda_5\mu_5 + \lambda_3\mu_5\lambda_5 + \lambda_4\lambda_5\lambda_5 + \lambda_1\mu_7\mu_1 + \lambda_2\lambda_7\mu_1 + \lambda_3\mu_7\lambda_1 + \lambda_4\lambda_7\lambda_1 \\
& + \lambda_5\mu_1\mu_7 + \lambda_6\lambda_1\mu_7 + \lambda_7\mu_1\lambda_7 + \lambda_8\lambda_1\lambda_7 + \lambda_5\mu_3\mu_3 + \lambda_6\lambda_3\mu_3 + \lambda_7\mu_3\lambda_3 + \lambda_8\lambda_3\lambda_3 \\
f_{90} = {} & \lambda_1\mu_5\mu_6 + \lambda_2\lambda_5\mu_6 + \lambda_3\mu_5\lambda_6 + \lambda_4\lambda_5\lambda_6 + \lambda_1\mu_7\mu_2 + \lambda_2\lambda_7\mu_2 + \lambda_3\mu_7\lambda_2 + \lambda_4\lambda_7\lambda_2 \\
& + \lambda_5\mu_2\mu_7 + \lambda_6\lambda_2\mu_7 + \lambda_7\mu_2\lambda_7 + \lambda_8\lambda_2\lambda_7 + \lambda_5\mu_4\mu_3 + \lambda_6\lambda_4\mu_3 + \lambda_7\mu_4\lambda_3 + \lambda_8\lambda_4\lambda_3 \\
f_{91} = {} & \lambda_1\mu_5\mu_7 + \lambda_2\lambda_5\mu_7 + \lambda_3\mu_5\lambda_7 + \lambda_4\lambda_5\lambda_7 + \lambda_1\mu_7\mu_3 + \lambda_2\lambda_7\mu_3 + \lambda_3\mu_7\lambda_3 + \lambda_4\lambda_7\lambda_3 \\
& + \lambda_5\mu_1\mu_5 + \lambda_6\lambda_1\mu_5 + \lambda_7\mu_1\lambda_5 + \lambda_8\lambda_1\lambda_5 + \lambda_5\mu_3\mu_1 + \lambda_6\lambda_3\mu_1 + \lambda_7\mu_3\lambda_1 + \lambda_8\lambda_3\lambda_1 \\
f_{92} = {} & \lambda_1\mu_5\mu_8 + \lambda_2\lambda_5\mu_8 + \lambda_3\mu_5\lambda_8 + \lambda_4\lambda_5\lambda_8 + \lambda_1\mu_7\mu_4 + \lambda_2\lambda_7\mu_4 + \lambda_3\mu_7\lambda_4 + \lambda_4\lambda_7\lambda_4 \\
& + \lambda_5\mu_2\mu_5 + \lambda_6\lambda_2\mu_5 + \lambda_7\mu_2\lambda_5 + \lambda_8\lambda_2\lambda_5 + \lambda_5\mu_4\mu_1 + \lambda_6\lambda_4\mu_1 + \lambda_7\mu_4\lambda_1 + \lambda_8\lambda_4\lambda_1 \\
f_{93} = {} & \lambda_1\mu_6\mu_5 + \lambda_2\lambda_6\mu_5 + \lambda_3\mu_6\lambda_5 + \lambda_4\lambda_6\lambda_5 + \lambda_1\mu_8\mu_1 + \lambda_2\lambda_8\mu_1 + \lambda_3\mu_8\lambda_1 + \lambda_4\lambda_8\lambda_1 \\
& + \lambda_5\mu_1\mu_8 + \lambda_6\lambda_1\mu_8 + \lambda_7\mu_1\lambda_8 + \lambda_8\lambda_1\lambda_8 + \lambda_5\mu_3\mu_4 + \lambda_6\lambda_3\mu_4 + \lambda_7\mu_3\lambda_4 + \lambda_8\lambda_3\lambda_4 \\
f_{94} = {} & \lambda_1\mu_6\mu_6 + \lambda_2\lambda_6\mu_6 + \lambda_3\mu_6\lambda_6 + \lambda_4\lambda_6\lambda_6 + \lambda_1\mu_8\mu_2 + \lambda_2\lambda_8\mu_2 + \lambda_3\mu_8\lambda_2 + \lambda_4\lambda_8\lambda_2 \\
& + \lambda_5\mu_2\mu_8 + \lambda_6\lambda_2\mu_8 + \lambda_7\mu_2\lambda_8 + \lambda_8\lambda_2\lambda_8 + \lambda_5\mu_4\mu_4 + \lambda_6\lambda_4\mu_4 + \lambda_7\mu_4\lambda_4 + \lambda_8\lambda_4\lambda_4 \\
f_{95} = {} & \lambda_1\mu_6\mu_7 + \lambda_2\lambda_6\mu_7 + \lambda_3\mu_6\lambda_7 + \lambda_4\lambda_6\lambda_7 + \lambda_1\mu_8\mu_3 + \lambda_2\lambda_8\mu_3 + \lambda_3\mu_8\lambda_3 + \lambda_4\lambda_8\lambda_3 \\
& + \lambda_5\mu_1\mu_6 + \lambda_6\lambda_1\mu_6 + \lambda_7\mu_1\lambda_6 + \lambda_8\lambda_1\lambda_6 + \lambda_5\mu_3\mu_2 + \lambda_6\lambda_3\mu_2 + \lambda_7\mu_3\lambda_2 + \lambda_8\lambda_3\lambda_2 \\
f_{96} = {} & \lambda_1\mu_6\mu_8 + \lambda_2\lambda_6\mu_8 + \lambda_3\mu_6\lambda_8 + \lambda_4\lambda_6\lambda_8 + \lambda_1\mu_8\mu_4 + \lambda_2\lambda_8\mu_4 + \lambda_3\mu_8\lambda_4 + \lambda_4\lambda_8\lambda_4 \\
& + \lambda_5\mu_2\mu_6 + \lambda_6\lambda_2\mu_6 + \lambda_7\mu_2\lambda_6 + \lambda_8\lambda_2\lambda_6 + \lambda_5\mu_4\mu_2 + \lambda_6\lambda_4\mu_2 + \lambda_7\mu_4\lambda_2 + \lambda_8\lambda_4\lambda_2
\end{aligned}
$$

$$f_{97} = -\mu_1\mu_1 - \mu_2\lambda_1 + \lambda_1\mu_2\mu_2 + \lambda_2\lambda_2\mu_2 + \lambda_3\mu_2\lambda_2 + \lambda_4\lambda_2\lambda_2 + \lambda_1\mu_4\mu_6 + \lambda_2\lambda_4\mu_6 + \lambda_3\mu_4\lambda_6 + \lambda_4\lambda_4\lambda_6$$
$$f_{98} = -\mu_1\mu_2 - \mu_2\lambda_2 + \lambda_1\mu_1\mu_2 + \lambda_2\lambda_1\mu_2 + \lambda_3\mu_1\lambda_2 + \lambda_4\lambda_1\lambda_2 + \lambda_1\mu_3\mu_6 + \lambda_2\lambda_3\mu_6 + \lambda_3\mu_3\lambda_6 + \lambda_4\lambda_3\lambda_6$$
$$f_{99} = -\mu_1\mu_3 - \mu_2\lambda_3 + \lambda_1\mu_2\mu_1 + \lambda_2\lambda_2\mu_1 + \lambda_3\mu_2\lambda_1 + \lambda_4\lambda_2\lambda_1 + \lambda_1\mu_4\mu_5 + \lambda_2\lambda_4\mu_5 + \lambda_3\mu_4\lambda_5 + \lambda_4\lambda_4\lambda_5$$
$$f_{100} = -\mu_1\mu_4 - \mu_2\lambda_4 + \lambda_1\mu_1\mu_1 + \lambda_2\lambda_1\mu_1 + \lambda_3\mu_1\lambda_1 + \lambda_4\lambda_1\lambda_1 + \lambda_1\mu_3\mu_5 + \lambda_2\lambda_3\mu_5 + \lambda_3\mu_3\lambda_5 + \lambda_4\lambda_3\lambda_5$$
$$f_{101} = -\mu_1\mu_5 - \mu_2\lambda_5 + \lambda_5\mu_2\mu_2 + \lambda_6\lambda_2\mu_2 + \lambda_7\mu_2\lambda_2 + \lambda_8\lambda_2\lambda_2 + \lambda_5\mu_4\mu_6 + \lambda_6\lambda_4\mu_6 + \lambda_7\mu_4\lambda_6 + \lambda_8\lambda_4\lambda_6$$
$$f_{102} = -\mu_1\mu_6 - \mu_2\lambda_6 + \lambda_5\mu_1\mu_2 + \lambda_6\lambda_1\mu_2 + \lambda_7\mu_1\lambda_2 + \lambda_8\lambda_1\lambda_2 + \lambda_5\mu_3\mu_6 + \lambda_6\lambda_3\mu_6 + \lambda_7\mu_3\lambda_6 + \lambda_8\lambda_3\lambda_6$$
$$f_{103} = -\mu_1\mu_7 - \mu_2\lambda_7 + \lambda_5\mu_2\mu_1 + \lambda_6\lambda_2\mu_1 + \lambda_7\mu_2\lambda_1 + \lambda_8\lambda_2\lambda_1 + \lambda_5\mu_4\mu_5 + \lambda_6\lambda_4\mu_5 + \lambda_7\mu_4\lambda_5 + \lambda_8\lambda_4\lambda_5$$
$$f_{104} = -\mu_1\mu_8 - \mu_2\lambda_8 + \lambda_5\mu_1\mu_1 + \lambda_6\lambda_1\mu_1 + \lambda_7\mu_1\lambda_1 + \lambda_8\lambda_1\lambda_1 + \lambda_5\mu_3\mu_5 + \lambda_6\lambda_3\mu_5 + \lambda_7\mu_3\lambda_5 + \lambda_8\lambda_3\lambda_5$$
$$f_{105} = -\mu_3\mu_1 - \mu_4\lambda_1 + \lambda_1\mu_2\mu_4 + \lambda_2\lambda_2\mu_4 + \lambda_3\mu_2\lambda_4 + \lambda_4\lambda_2\lambda_4 + \lambda_1\mu_4\mu_8 + \lambda_2\lambda_4\mu_8 + \lambda_3\mu_4\lambda_8 + \lambda_4\lambda_4\lambda_8$$
$$f_{106} = -\mu_3\mu_2 - \mu_4\lambda_2 + \lambda_1\mu_1\mu_4 + \lambda_2\lambda_1\mu_4 + \lambda_3\mu_1\lambda_4 + \lambda_4\lambda_1\lambda_4 + \lambda_1\mu_3\mu_8 + \lambda_2\lambda_3\mu_8 + \lambda_3\mu_3\lambda_8 + \lambda_4\lambda_3\lambda_8$$
$$f_{107} = -\mu_3\mu_3 - \mu_4\lambda_3 + \lambda_1\mu_2\mu_3 + \lambda_2\lambda_2\mu_3 + \lambda_3\mu_2\lambda_3 + \lambda_4\lambda_2\lambda_3 + \lambda_1\mu_4\mu_7 + \lambda_2\lambda_4\mu_7 + \lambda_3\mu_4\lambda_7 + \lambda_4\lambda_4\lambda_7$$
$$f_{108} = -\mu_3\mu_4 - \mu_4\lambda_4 + \lambda_1\mu_1\mu_3 + \lambda_2\lambda_1\mu_3 + \lambda_3\mu_1\lambda_3 + \lambda_4\lambda_1\lambda_3 + \lambda_1\mu_3\mu_7 + \lambda_2\lambda_3\mu_7 + \lambda_3\mu_3\lambda_7 + \lambda_4\lambda_3\lambda_7$$
$$f_{109} = -\mu_3\mu_5 - \mu_4\lambda_5 + \lambda_5\mu_2\mu_4 + \lambda_6\lambda_2\mu_4 + \lambda_7\mu_2\lambda_4 + \lambda_8\lambda_2\lambda_4 + \lambda_5\mu_4\mu_8 + \lambda_6\lambda_4\mu_8 + \lambda_7\mu_4\lambda_8 + \lambda_8\lambda_4\lambda_8$$
$$f_{110} = -\mu_3\mu_6 - \mu_4\lambda_6 + \lambda_5\mu_1\mu_4 + \lambda_6\lambda_1\mu_4 + \lambda_7\mu_1\lambda_4 + \lambda_8\lambda_1\lambda_4 + \lambda_5\mu_3\mu_8 + \lambda_6\lambda_3\mu_8 + \lambda_7\mu_3\lambda_8 + \lambda_8\lambda_3\lambda_8$$
$$f_{111} = -\mu_3\mu_7 - \mu_4\lambda_7 + \lambda_5\mu_2\mu_3 + \lambda_6\lambda_2\mu_3 + \lambda_7\mu_2\lambda_3 + \lambda_8\lambda_2\lambda_3 + \lambda_5\mu_4\mu_7 + \lambda_6\lambda_4\mu_7 + \lambda_7\mu_4\lambda_7 + \lambda_8\lambda_4\lambda_7$$
$$f_{112} = -\mu_3\mu_8 - \mu_4\lambda_8 + \lambda_5\mu_1\mu_3 + \lambda_6\lambda_1\mu_3 + \lambda_7\mu_1\lambda_3 + \lambda_8\lambda_1\lambda_3 + \lambda_5\mu_3\mu_7 + \lambda_6\lambda_3\mu_7 + \lambda_7\mu_3\lambda_7 + \lambda_8\lambda_3\lambda_7$$

$$f_{113} = -\mu_5\mu_1 - \mu_6\lambda_1 + \lambda_1\mu_6\mu_2 + \lambda_2\lambda_6\mu_2 + \lambda_3\mu_6\lambda_2 + \lambda_4\lambda_6\lambda_2 + \lambda_1\mu_8\mu_6 + \lambda_2\lambda_8\mu_6 + \lambda_3\mu_8\lambda_6 + \lambda_4\lambda_8\lambda_6$$
$$f_{114} = -\mu_5\mu_2 - \mu_6\lambda_2 + \lambda_1\mu_5\mu_2 + \lambda_2\lambda_5\mu_2 + \lambda_3\mu_5\lambda_2 + \lambda_4\lambda_5\lambda_2 + \lambda_1\mu_7\mu_6 + \lambda_2\lambda_7\mu_6 + \lambda_3\mu_7\lambda_6 + \lambda_4\lambda_7\lambda_6$$
$$f_{115} = -\mu_5\mu_3 - \mu_6\lambda_3 + \lambda_1\mu_6\mu_1 + \lambda_2\lambda_6\mu_1 + \lambda_3\mu_6\lambda_1 + \lambda_4\lambda_6\lambda_1 + \lambda_1\mu_8\mu_5 + \lambda_2\lambda_8\mu_5 + \lambda_3\mu_8\lambda_5 + \lambda_4\lambda_8\lambda_5$$
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Non-commutative and non-anticommutative

We have 128 different polynomials in $\mathbb{K}[\lambda_1, \ldots, \lambda_8, \mu_1, \ldots, \mu_8]$
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Theorem

Let $\mathbb{K}$ be a field of characteristic zero. If $\mathcal{V}$ is a (LACC) variety of non-associative algebras without any identity of order 2, then $\mathcal{V}$ is abelian.

Subvarieties of Lie and *n*-arity

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Then it is just enough to compute a Gröbner basis for the prime divisors of this n.

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Both varieties $\mathtt{Lie}_{\mathbb{K}}$ and $\mathtt{qLie}_{\mathbb{K}}$ are (LACC).

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Preprint, arXiv:1701.05493. 2017.

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