Lectures
Conjugate harmonicity in Euclidean space

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Let $\Omega \subset \mathbb{C}$ be open and simply connected and let $f = u + i\ v : \Omega \to \mathbb{C}$ be a C_1-function in Ω. As is well known, the following assertions are then equivalent:

(i) f is holomorphic in Ω, i.e. $\partial_{\bar{z}} f = 0$ in Ω where $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$

(ii) The pair (u, v) is a conjugate harmonic pair in Ω, i.e. (u, v) satisfies in Ω the Cauchy-Riemann system

$$\begin{cases} \partial_x u - \partial_y v &= 0 \\ \partial_y u + \partial_x v &= 0 \end{cases}$$

(iii) The 1-form $\omega = udx + vdy$ satisfies in Ω the Hodge-de Rham system

$$\begin{cases} d\omega &= 0 \\ d^*\omega &= 0 \end{cases}$$

(iv) There exists U, $\mathbb{R}$-valued and harmonic in Ω such that $f = \partial_z U$ where $\partial_z = \dfrac{1}{2}(\partial_x - i\partial_y)$

(v) f admits a holomorphic primitive F in Ω, i.e. $f = \dfrac{dF}{dz} = \partial_z F$

Harmonic conjugates play an important role in harmonic analysis in the plane, in particular in studying Hardy spaces for the unit circle and the upper half plane.

In higher-dimensional Euclidean space $\mathbb{R}^{m+1}$, Clifford analysis - a function theory for the Dirac operator ∂_x, or, equivalently for the Cauchy-Riemann operator D_x in $\mathbb{R}^{m+1}$- generalizes a lot of basic results of complex analysis in the plane.

Let $\mathbb{R}^{0,m+1}$ be the vector space $\mathbb{R}^{m+1}$ provided with a quadratic form of signature (0,m+1); let $e = (e_0, e_1, ..., e_m)$ be an orthonormal basis for $\mathbb{R}^{0,m+1}$; let $\mathbb{R}_{0,m+1}$ be the real Clifford algebra constructed over $\mathbb{R}^{0,m+1}$ and let $(e_A : A \subset \{0,1,...,m\})$ be the standard basis for $\mathbb{R}_{0,m+1}$. Putting for $A = \{i_1,...,i_r\} \subset \{0,1,...,m\}, e_A = e_{i_1}...e_{i_r}$ with $e_\emptyset = 1$, the identity element of $\mathbb{R}_{0,m+1}$, we then have that

$$\mathbb{R}_{0,m+1} = \sum_{r=0}^{m+1} \oplus \, \mathbb{R}_{0,m+1}^{(r)},$$

where

$$\mathbb{R}_{0,m+1}^{(r)} = span_{\mathbb{R}}(e_A : |A| = r)$$

is the space of so-called r-vectors in $\mathbb{R}_{0,m+1}$.

Furthermore, let $\Omega \subset \mathbb{R}^{m+1}$ be open and let $F : \Omega \to \mathbb{R}_{0,m+1}$ be a C_1-function in Ω. Then F is called left (resp. right) monogenic in Ω if

$$\partial_x F = 0, \text{ resp. } F\partial_x = 0 \text{ in } \Omega \tag{1}$$

Hereby $\partial_x = \sum_{i=0}^{m} e_i \partial_{x_i}$ is the Dirac operator in $\mathbb{R}^{m+1}$.
Putting $D_x = \overline{e}_0 \, \partial_x = \partial_{x_0} + \overline{e}_0 \, \partial_{\underline{x}}$ where $\overline{e}_0 = -e_0$ and $\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j}$ is the Dirac operator in $\mathbb{R}^m$, D_x is then called the Cauchy-Riemann operator in $\mathbb{R}^{m+1}$.
Clearly

$$\partial_x F = 0 \Longleftrightarrow D_x F = 0$$

Moreover, by means of the decomposition

$$\mathbb{R}_{0,m+1} = \mathbb{R}_{0,m} \oplus \overline{e}_0 \, \mathbb{R}_{0,m}$$

where $\mathbb{R}_{0,m}$ is the Clifford algebra generated inside $\mathbb{R}_{0,m+1}$ by the orthonormal basis $\underline{e} = (e_1,...,e_m)$ of the quadractic space $\mathbb{R}^{0,m}$, we have that $F : \Omega \to \mathbb{R}_{0,m+1}$ may thus be composed as

$$F = U + \overline{e}_0 \, V$$

where U and V are $\mathbb{R}_{0,m}$-valued.
We then have in Ω:

$$D_x F = 0 \Longleftrightarrow \begin{cases} \partial_{x_0} U + \partial_{\underline{x}} V & = & 0 \\ \partial_{\underline{x}} U \; + \partial_{x_0} V & = & 0 \end{cases} \tag{2}$$

A pair (U, V) of $\mathbb{R}_{0,m}$-valued functions in Ω satisfying (2) is called conjugate harmonic in Ω.
Important examples of left monogenic functions in Ω are given by 1-vector valued

functions $F = \sum_{i=0}^m e_i F_i$ satisfying in Ω

$$\partial_x F = 0 \Longleftrightarrow \left\{ \begin{array}{rcl} \displaystyle\sum_{i=0}^m \frac{\partial F_i}{\partial x_i} & = & 0 \\ \displaystyle\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i} & = & 0 \quad i \neq j, \ i,j = 0,1,...,m \end{array} \right. \tag{3}$$

Putting $\overrightarrow{F} = (F_0, F_1, ..., F_m)$ then the system (3) is clearly equivalent to the Riesz-system

$$\left\{ \begin{array}{rcl} \text{div } \overrightarrow{F} & = & 0 \\ \text{curl } \overrightarrow{F} & = & 0 \end{array} \right. \tag{4}$$

A set $\overrightarrow{F} = (F_0, F_1, ..., F_m)$ satisfying (4) was called by Stein-Weiss a system of conjugate harmonic functions in Ω. This is equivalent to saying that the $\mathbb{R} \oplus \ \overline{e}_0 \ \mathbb{R}^{0,m}$-valued function

$$\begin{aligned} F^* &= F_0 + \overline{e}_0 \sum_{j=1}^m e_j(-F_j) \\ &= U + \overline{e}_0 V \end{aligned}$$

satisfies $D_x F^* = 0$ in Ω, i.e. (U, V) is a conjugate harmonic pair.
Considering the smooth 1-form

$$\omega = \sum_{i=0}^m \omega_i \ dx^i$$

where $\omega_i = F_i, i = 0, 1, ..., m$, then (4) is equivalent to saying that ω satisfies the Hodge-de Rham system

$$\left\{ \begin{array}{rcl} d\omega & = & 0 \\ d^*\omega & = & 0 \end{array} \right. ,$$

i.e. ω is a harmonic vector field in Ω.

The aim of this series of lectures is to discuss the relationship between monogenic functions, conjugate harmonicity and its applications, a.o. to the construction of primitives of monogenic functions; to the construction of bases for the space of homogeneous monogenic $\mathbb{R} \oplus \overline{e}_0\mathbb{R}^{0,m}$-valued polynomials; to the structure of the Hardy space $H^2(\mathbb{R}^{m+1}_+)$, where $\mathbb{R}^{m+1}_+ = \{x = (x_0, x_1, ..., x_m) \in \mathbb{R}^{m+1} : \ x_0 > 0\}$.

Contents

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