COROLLARY 8. $\|S_A\|_\infty = \left\| S_{\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}} \right\|_w \quad (A \in \mathbb{M}_n).$

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GENERALIZED INVARIANT FACTORS

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SYNOPSIS

Introduction

In the last 10–12 years, a lot of work has been done concerning invariant factors of matrices, mainly by R. C. Thompson (Santa Barbara, U.S.A.) and E. Marques de Sá (Coimbra, Portugal). Many things are now known about

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invariant factors of matrices and submatrices, sums of matrices, products of matrices, etc., the results often taking the form of divisibility relations. (See below for details and references.)

The purpose of this talk is simply to call attention to the fact that many of these results, usually presented for matrices over principal-ideal domains, actually hold for larger classes of rings.

Elementary-Divisor Domains and Beyond

We look first at elementary-divisor domains, a class introduced by Kaplansky in [2]. By definition, these are domains where Smith's diagonalization theorem holds for every matrix. An example of an elementary-divisor domain which is not a principal-ideal domain is the ring of analytic functions $H(\Omega)$, Ω an open connected subset of the complex plane. Elementary-divisor domains are not necessarily unique-factorization domains, so proofs that use localization at a prime do not carry over from principal-ideal domains.

Extension of results to elementary-divisor domains is doubly interesting in that, using a device due to Krull, it often allows extension to even larger classes of rings. We give a brief description of this technique.

An integral domain V is a *valuation domain* (in its field of quotients) if, for all $a, b \in V$, either $a \mid b$ or $b \mid a$.

Let R be an integral domain, K its field of quotients. The integral closure of R (in K) is

$$\overline{R} = \{q \in K : f(q) = 0 \text{ for some monic } f(x) \in R[x]\}$$

R is *integrally closed* if $\overline{R} = R$. Example: $\mathbf{Z}$.

THEOREM [3]. *$\overline{R}$ equals the intersection of all valuation domains that contain R.*

Therefore, if R is integrally closed, a divisibility relation involving elements of R holds in R if it holds in every valuation domain V containing R.

Now it is trivial that every valuation domain is an elementary-divisor domain (diagonalization is easy, since—up to associates—divisibility is a total order). Hence divisibility relations proved for arbitrary elementary-divisor domains (or for arbitrary valuation domains) may be used to obtain statements valid for integrally closed domains. This technique was used in [2] to prove some simple divisibility relations for invariant factors of matrices over integrally closed domains. Here we merely remark that the same trick works for many relations discovered recently.

Invariant Factors

If a matrix A over an integral domain is equivalent to a diagonal matrix where each diagonal entry divides the following, these entries are, by definition, the invariant factors of A. The invariant factors of a matrix A will be denoted by $s_1(A) \mid s_2(A) \mid \cdots$ (we add an infinite tail of zeros). Their uniqueness is guaranteed by their expressions as quotients of determinantal divisors. These expressions can in turn be used to define the invariant factors for matrices not equivalent to a diagonal. For this to work, we must require that in the underlying domain any finite set of elements have a greatest common divisor (not necessarily expressible as a linear combination of the elements). Rings with this property are usually called *gcd domains*. (Bourbaki calls them pseudo-Bézoutian.) They are easily seen to be integrally closed. Elementary-divisor domains and unique-factorization domains all are gcd domains.

Results

We list some divisibility relations concerning invariant factors of matrices which are true over elementary-divisor domains. Most of these can be shown, with simple proofs that use the argument described above, to hold for invariant factors of matrices over gcd domains.

(1) The very fact that, for all A and k, $s_k(A)$ divides $s_{k+1}(A)$ is an example of a divisibility relation which extends to matrices over arbitrary gcd domains after its usual proof employing the reduction to Smith normal form of matrices over elementary-divisor domains.

(2) If A' $(m' \times n')$ is a submatrix of A $(m \times n)$, then, for all i,

$$s_i(A) \mid s_i(A') \mid s_{i+(m-m')+(n-n')}(A)$$

(the so-called interlacing "inequalities" [6, 7]). This can be proved in several ways using only the existence of the Smith normal form, so it holds for matrices over elementary-divisor domains (and it extends to matrices over gcd domains). One of the simplest proofs uses the following characterization of invariant factors:

$$s_k(A) = \operatorname{lcm}\{\gcd(a_{ij} - x_{ij}) : \operatorname{rank}(X) < k\}$$

([4]; there the result was stated for principal-ideal domains, but the same proof works for elementary-divisor domains).

(3) Concerning invariant factors of sums of matrices, we have the following relation:

$$\gcd(s_i(A), s_j(B)) \mid s_{i+j-1}(A+B)$$

for all i, j. The proof of this is trivial using the above characterization of invariant factors, so this relation extends to matrices over arbitrary elementary-divisor domains. (The original proof [8] uses localization at a prime.)

(4) Invariant factors of products of matrices have been extensively studied. For $n \times n$ A and B, known relations have the form

$$s_{i_1}(A) \cdots s_{i_t}(A) s_{j_1}(B) \cdots s_{j_t}(B) \mid s_{k_1}(AB) \cdots s_{k_t}(AB), \tag{P}$$

where $1 \leqslant t \leqslant n$, $1 \leqslant i_1 \leqslant \cdots \leqslant i_t \leqslant n$, $1 \leqslant j_1 \leqslant \cdots \leqslant j_t \leqslant n$, $1 \leqslant k_1 \leqslant \cdots \leqslant k_t \leqslant n$. The problem is to find all the "right" sequences i, j, k.

A very general description of allowed sequences (suspected to be the complete answer) is in [9], using the language of Young tableaux and Littlewood-Richardson sequences. For the (very intricate) proof to work, the ring must be a principal-ideal domain.

An important corollary of that work is that (P) holds when $k_u = i_u + j_u - u$, $1 \leqslant u \leqslant t$ (the "standard" inequalities). For $t = 1$, this gives the well-known relation

$$s_i(A) s_j(B) \mid s_{i+j-1}(AB).$$

Some results on this problem that hold for arbitrary elementary-divisor domains (although stated for principal-ideal domains) are contained in a 1978 unpublished manuscript by E. Marques de Sá [5]. The main theorem is:

Suppose (P) *holds for all* $n\times n$ A *and* B, *and let* $u,v\in\{1,\dots,t+1\}$, $w\in\{1,\dots,t\}$. *If* $i_u+j_v\geqslant k_{w-1}+k_t+2$, *then*

$$s_{i_1}(A)\cdots s_{i_{u-1}}(A)s_{i_u+1}(A)\cdots s_{i_t+1}(A)s_{j_1}(B)\cdots$$
$$\times s_{j_{v-1}}(B)s_{j_v+1}(B)\cdots s_{j_t+1}(B)$$
$$\mid s_{k_1}(AB)\cdots s_{k_{w-1}}(AB)s_{k_w+1}(AB)\cdots s_{k_t+1}(AB)$$

holds for all $(n+1)\times(n+1)$ A *and* B (*with* $k_0=0$, $i_{t+1}=j_{t+1}=k_t+1$ *by definition*).

The reader should note how this can be used to obtain new relations from known ones. (Compare with [1].) Example: Starting with $n=t$ and the sequences $i=j=k=(1,\dots,t)$ (obviously right) and applying the result several times, we obtain the standard inequalities, which therefore hold for arbitrary elementary-divisor domains, and also for arbitrary gcd domains.

Open Problems

It is natural to ask for other instances of divisibility relations whose proofs can be changed so that they work for matrices over arbitrary elementary-divisor domains. An obvious (but presumably intractable) candidate is Thompson's result on the invariant factors of a product [9, p. 431].

Different questions arise when we consider "inverse" problems. For example, it is well known that the interlacing inequalities are the only general relations connecting the invariant factors of matrices and submatrices, in the sense that, given elements satisfying those relations, there exist a matrix and a submatrix with those elements as invariant factors. This last assertion is easily proved by induction when the ring in question is an elementary-divisor domain. Over a gcd domain the same problem is, to my knowledge, open.

Let us see an example of an inverse statement for which the extension from principal-ideal domains to elementary-divisor domains already presents a challenge. If A and B are $n\times n$, it follows from the standard inequalities that

$$\operatorname{lcm}\{s_i(A)s_{k-i+1}(B):1\leqslant i\leqslant k\}\mid s_k(AB)\mid\gcd\{s_i(A)s_{n-i+k}(B):k\leqslant i\leqslant n\}.$$

If the ring is a principal-ideal domain, the converse is true: given $a_1\mid\cdots\mid a_n$, $b_1\mid\cdots\mid b_n$, and c, if

$$\operatorname{lcm}\{a_ib_{k-i+1}:1\leqslant i\leqslant k\}\mid c\mid\gcd\{a_ib_{n-i+k}:k\leqslant i\leqslant n\},$$

then there exist A and B $n\times n$ with invariant factors $a_1,\dots,a_n$ and $b_1,\dots,b_n$, respectively, and such that $s_k(AB)=c$ (J. F. Queiró and E. Marques de Sá, to be submitted). Question: can this be proved for elementary-divisor domains or, harder still, for gcd domains?

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