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# JOSÉ ANASTÁCIO DA CUNHA:
# AN ASSESSMENT

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*I shall devote all my efforts to bring light into the immense obscurity that today reigns in Analysis. It so lacks any plan or system, that one is really astonished that there are so many people who devote themselves to it – and, still worse, it is absolutely devoid of any rigour.*

Niels Abel, 1826

*The greatest mathematicians of the 18th century had not the habit of definition.*

G. H. Hardy, 1949

## Introduction – Portugal in the mid-18th century

In the first half of the 18th century, education and science in Portugal were influenced, as they had been throughout the previous century, by the Jesuit Order. The Jesuits' presence was strong mainly in primary and secondary studies, in which they used the traditional teaching associated with the Order. But they did not have a monopoly. In particular, there was increasing activity by the Oratorians, specially in scientific teaching and regular scientific demonstrations. This brought them grants from King John V and the interest of King Joseph. The period was one of enthusiasm for scientific experiments and observations, with growing emphasis on experience as guide and arbiter of all statements concerning the sciences.

In May 1762, towards the end of the Seven Years' War, Spain (in alliance with France) invaded Portugal. During the summer of that year, Portugal received military help from England (troops, arms, and money). At the same time the Prussian Count of Schaumburg-Lippe was put in charge of reorganizing the Portuguese Army. After the peace in February 1763, Lippe's activity in Portugal continued.

The evolution of the Portuguese Army in the mid-1700s can be summarized in the following table, which shows numbers of men:

| | 1735 | 1762 | 1764 |
|---|---|---|---|
| Infantry | 24 000 | 40 000 | 20 688 |
| Cavalry | 7200 | 5838 | 5838 |
| Artillery | 1800 | 2160 | 2880 |

Lippe stayed in Portugal for long periods, first in 1762--1764 and later in 1767. Much was done under his direction concerning organization, fortification and the instruction of military personnel. Great attention was paid to the training of the officer corps, with emphasis on hierarchy, and high value put on technical knowledge. The impor-



tance of competent artillery units was recognized, with a substantial increase in their personnel.

At that time, political power in Portugal was in the hands of the Count of Oeiras, later Marquis of Pombal, the first minister of King Joseph. His handling of the reconstruction of Lisbon after the 1755 earthquake reinforced his authority, which he used to foster economic development as well as to fight against the higher nobility. He was the main force behind the expulsion of the Jesuits in 1759, which included the closure of the University of Évora.

In 1772, an extensive reform of the University of Coimbra was carried out under Pombal's supervision. Two new faculties, Mathematics and Philosophy (meaning the Natural Sciences), were created, and new statutes were enacted, with a strong emphasis on the importance of science in general and mathematics in particular.

The death of King Joseph in 1777 and the accession of his daughter Queen Mary I led to the fall of Pombal and the reversal of some of his policies.

## Life of José Anastácio da Cunha

José Anastácio da Cunha was born in Lisbon, on May 11, 1744. He was educated by the Oratorians until 1763, when he enlisted in the military. He became a lieutenant in the artillery regiment stationed at Valença, in the north of Portugal. This regiment had a number of foreign officers with whom Cunha established personal and intellectual relationships.

Around 1767 an interesting incident took place involving Cunha and Count Lippe, when Cunha corrected some errors in an official instruction manual. After an initial reaction of displeasure, Lippe recommended Cunha for promotion.

In 1773, Cunha was made a Professor at the University of Coimbra, following a letter from Pombal himself to the Rector, in which he cites recommendations by high-placed military officers. Cunha taught at the new Faculty of Mathematics until 1778. In July 1 of that year, he was arrested by the Inquisition, on charges of free thinking and religious heterodoxy, associated with his life in Valença and with his poetry (much later praised by the likes of Almeida Garrett and Fernando Pessoa). He was sentenced to house arrest at the Oratorians, in Lisbon, and forbidden ever to return to Valença or Coimbra.

He was pardoned on January 23, 1781, and spent the rest of his life teaching at the Casa Pia, a school for poor children. He died in Lisbon, on January 1, 1787, at the age of 42. All works by José Anastácio da Cunha were published posthumously. The complete list is as follows:

*Ensaio sobre as Minas (1767)*, Braga 1994
*Carta Physico-Mathematica (1769)*, Porto 1838
*Notícias Literárias de Portugal (1780)*, Lisboa 1966
*Principios Mathematicos*, Lisboa 1790
*Ensaio sobre Principios de Mechanica*, London 1807
*Principes Mathématiques*, Bordeaux 1811, Paris 1816
[*Poetical Works*], Porto 1839, Coimbra 1930, Porto 2001

Before continuing, let us briefly review an aspect of the history of Mathematics.



## Early history of the calculus

At the end of the 17th century, Newton and Leibniz, independently, created the differential and integral calculus. This extraordinary mathematical instrument allowed the treatment and solution of many mathematical and physical problems, foremost among which was the proof of Kepler's laws on the movement of the planets, starting from the law of universal gravitation.

The creation of the calculus initiated an era in which the work with the infinitely large and the infinitely small became common. The new "infinitary methods" were applied to all sorts of geometrical and physical problems. Throughout the 18th century, many mathematicians, including Leonhard Euler, the Bernoullis, d'Alembert and Lagrange, exploited multiple applications of the new techniques. It was a period of great advances and discoveries and of the development of new theories based on the calculus, also called mathematical analysis.

All this extraordinary progress was carried out without much concern for the rigorous foundations of the edifice under construction. It is true that the question of the foundations of analysis received the attention of some mathematicians and philosophers during the 18th century, who understood that the search for rigour was not a vain exercise: the legitimacy and certainty of the results are all the more necessary as the applications of the theory become wider and wider. The Berlin Academy of Sciences even established a prize, in 1784, for the best explanation of the concept of the infinite. (In 1786 the Academy published its decision, stating that none of the contestants had presented a satisfactory answer to the problem, and giving the prize to the Swiss Lhuilier for presenting the work which was closest to the contest's intentions.)

The Statutes of the reformed University of Coimbra, published in 1772, made a reference to these problems. In the section on the second year of the Mathematics studies course, we read, among other pertinent remarks:

> "The lecturer shall have great care in teaching the fundamental principles of the Differential Calculus in the easiest and most understandable way; showing with full distinction and clarity what is meant by fluxions or infinitesimal elements; and trying to free the students' minds of the equivocations in the explanation of those principles, proceeding from the shaky ideas of a dark Metaphysics, which led many authors to treat this Calculus from a disadvantageous point of view."

Only in the 19th century did a critical revision of the foundations of the calculus take place, in a search for rigour and precision in the main concepts and methods, especially the infinitary processes. These objectives came to be achieved with the so-called "arithmetization of Analysis", a programme of numeric and algebraic foundation involving names like Bolzano, Abel, Cauchy, Dedekind and Weierstrass. With the work of these and other mathematicians, great rigour was introduced in mathematical analysis, in the treatment of the infinitely large and the infinitely small, and in central concepts such as the very definition of real numbers.



## Features of Cunha's thought

In his ten years of military life, there are sure signs that Anastácio da Cunha studied mathematics regularly. The *Ensaio sobre as Minas* and the *Carta Fisico-Mathematica*, concerning scientific problems related to military activity, such as mine explosions and ballistics, are the only known works of his from this period. What is important about them is not so much their subject matter as the personality traits they reveal about the author, and which were to become manifest in his later works. There are essentially three of these traits: a critical mind (with contempt for authority arguments), a search for clarity and rigour, and a belief in the power of reason to base and organize knowledge. To this we should add a preference for concise expression and, reflecting the atmosphere of Cunha's intellectual education, the elevation of experience to the status of guide and arbiter of scientific inquiry. The clear distinction between the role of experience in the search for and establishment of physical laws, and the mathematical reasoning that applies to them and draws conclusions from them, was to be analysed with extraordinary rigour by Cunha in his later *Essay on the Principles of Mechanics*.

The first two works by Anastácio da Cunha show the author to have a free and acute mind, fearless in his criticisms of well-known authors and in his search for the best way to seek the truth and establish it on firm grounds. Though interesting, they would not by themselves give Cunha a place in the history of science. But the same disposition of mind would be present in a later work of much greater scope and ambition.

## The *Principios Mathematicos*

On the very day of his arrest, Cunha made a statement to his Inquisition interrogators:

> "The defendant said he hopes he will deserve mercy and pity, and be reconciled with the Church, so that he will be able to cry for his errors and do penance for them in the Congregation of the Oratory in Lisbon, where he wishes to retire, and hopes to do so when he is given back his freedom, not only for this main purpose, but also so that in that house he can be useful to the public and the State by publishing a work, which is the basis for all of Mathematics, on which he has been working for the last twelve years with the most assiduous and tireless dedication, and which he had already completed at the time of his imprisonment."

The book Cunha is talking about became the *Principios Mathematicos*, published in Lisbon three years after its author's death. The *Principios* has no preface or index and is made up of 21 chapters, or "Books", with no headings. The subjects covered are the following:

Book 1. Triangle geometry
Book 2. Circle geometry
Book 3. Proportions
Book 4. Elementary arithmetic
Book 5. Triangle similarity
Book 6. Solid geometry
Book 7. Circle geometry



Book 8. Elementary algebra
Book 9. Series and powers
Book 10. Algebraic equations
Book 11. Systems of equations
Book 12. Diophantine equations
Book 13. Geometrical problems
Book 14. Geometrical problems with conics
Book 15. Calculus
Book 16. Trigonometry
Book 17. Differential geometry
Book 18. Integration
Book 19. Differential equations
Book 20. Finite differences
Book 21. Miscellaneous problems

The work consists of a terse sequence of axioms--definitions-propositions-proofs, with insistence on rigour in all arguments, embodying an ambitious project to organize and expound mathematics on solid grounds. This kind of exposition is that of Euclid's *Elements*, the model for all deductive expositions. What is remarkable in José Anastácio da Cunha's *Principios*, and shows his ambition, is the material covered with such a model of exposition.

Although some of the subjects treated were then of recent discovery – showing the young author to be widely read and knowledgeable – the book is not a work of original research, and was seen by contemporaneous commentators as a banal compendium. But we repeat that José Anastácio da Cunha tries to define all concepts and justify all statements. It is therefore worth looking at the chapters dealing with subjects whose foundation at the time was nonexistent or controversial. The most important are Books 9 and 15.

## Book 9 of the *Principios Mathematicos*

Book 9 of the *Principios* deals with infinite series and some applications. Infinite series were extensively used throughout the 18th century, in close connection with the recently created calculus. This was done with great freedom and without particular concern for rigour or foundations problems, basically treating infinite series just like finite sums.

The first definition in Book 9 is the following:

> "Mathematicians call convergent a series whose terms are similarly determined, each one by the number of preceding terms, so that the series can always be continued, and eventually it is indifferent to continue it or not, because one may disregard without notable error the sum of how many terms one would wish to add to those already written or indicated; and the latter are indicated by writing &c. after the first two, or three, or how many one wishes; it is however necessary that the written terms show how the series might be continued, or that this be known through some other way."

So for Cunha the series

$$u_1 + u_2 + \cdots + u_n + \ldots$$



is convergent if is possible to reach a term, say the $n$th, after which it is indifferent to continue the series or not, because one may, for every $p$, disregard the sum

$$u_{n+1} + u_{n+2} + \ldots + u_{n+p}$$

without notable error.

There was no precedent at that time for a definition like this. The closest attempt may be the following one, by Euler, in 1734:

> "Series, quae in infinitum continuata summam habet finitam, etiamsi ea duplo longius continuetur, nullum accipiet augmentum, sed id, quod post infinitum adiicitur cogitacione, re vera erit infinite parvum. Nisi enim hoc ita se haberet, summa seriei etsi in infinitum continuatae non esset determinata et propterea non finita. Ex quo consequitur, si id, quod ex continuatione ultra terminum infinitesimum oritur, sit finitae magnitudinis, summam seriei necessario infinitam esse debere. Ex hoc ergo principio iudicare poterimus, utrum seriei cuiusque propositae summa sit infinita an finita."

In symbols, this is the same as saying that $\Sigma\, u_n$ is convergent if and only if $u_{N+1} + u_{N+2} + \ldots$ is infinitesimal when $N$ is infinite.

Cunha's definition is precisely the same as what later became known as the Bolzano-Cauchy criterion for convergence. This is confirmed by the use he makes of it immediately afterwards, in Proposition 1, by proving with complete correctness, and in a style typical of the 19th-

-century critical revision, the convergence of the geometric series

$$A + B + B\frac{B}{A} + B\frac{B}{A} \times \frac{B}{A} + B\frac{B}{A} \times \frac{B}{A} \times \frac{B}{A} + \&c.$$

when $A > B$. The proof uses Axiom 1 of Book 3, which says that "Given two different magnitudes, some multiple of the smaller will exceed the larger."

Cunha's is an "internal" criterion, in the sense that it involves only the terms of the series, without reference to the complete sum, whose concept remains implicit: if from a certain term onwards it is indifferent to continue the series or not, this is because the added terms yield an approximation (with prescribed maximum error) of the total sum.

Two Corollaries follow. The first states that the series

$$1 + a + \frac{aa}{2} + \frac{aaa}{2 \times 3} + \frac{aaaa}{2 \times 3 \times 4} + \frac{aaaaa}{2 \times 3 \times 4 \times 5} + \&c.$$

is convergent whatever the number $a$, and the second says that the series

$$a + \frac{1}{3}\,aaa + \frac{1}{5}\,aaaaa + \&c.$$

is convergent when $a < 1$. The sum of the first of these two series is what today we call the *exponential* of the number $a$.

Next comes another high point in this chapter, Definition 2:



"Let $a$ and $b$ represent arbitrary numbers, and let $c$ be the number that makes

$$1 + c + \frac{cc}{2} + \frac{ccc}{2 \times 3} + \frac{cccc}{2 \times 3 \times 4} + \&c. = a;$$

the expression $a^b$ shall mean a number

$$1 + bc + \frac{bbcc}{2} + \frac{bbbccc}{2 \times 3} + \frac{bbbbcccc}{2 \times 3 \times 4} + \&c.;$$

and the number $a^b$ shall be called the power of $a$ to the exponent $b$; or the root of $a$ indicated by the exponent $\frac{1}{b}$, denoted by $\sqrt[\frac{1}{b}]{a}$ (...)."

Immediately after this definition, Cunha proves, in Proposition 2, that for all positive $a$ there exists $c$ satisfying the first condition of the definition: $c$ is the sum of the series

$$2\left(\frac{a-1}{a+1} + \frac{1}{3}\;\frac{a+1}{a+1}\;\frac{a-1}{a-1}\quad\frac{a-1}{a+1} + \&c.\right)$$

which, by the second Corollary to Proposition 1, is convergent when $a > 0$. The number $c$ is what we call the *logarithm* of $a$.

Later results in Book 9, proved from Definition 2, include the following:

$$a^b a^c = a^{b+c}$$

$$a^{b-c} = \frac{a^b}{a^c}$$

$$a^0 = 1$$

$$a^{-c} = \frac{1}{a^b}$$

$$(a^b)^c = a^{bc}$$

In Proposition 7 Cunha proves the following:

"Let $A$ denote the preceding term. Then one has

$$(1+Q)^n = 1 + nQ + \frac{n-1}{2}AQ + \frac{n-2}{3}AQ + \frac{n-3}{4}AQ + \&c.$$

as long as, when $n$ is not a positive integer, we have $Q<1$."

This is the binomial series, with the correct conditions for convergence.

In Book 16, the definition is even applied to the case of imaginary exponents. Proposition 17 in that chapter states that

$$re^{\frac{z}{r}\sqrt{-1}} = \cos z + (\sqrt{-1}) \text{ sen } z$$

The series for sen $z$ and cos $z$ were previously obtained, with the comment: "It is easily shown that these series converge, for any $z$."



The definition of power proposed by José Anastácio da Cunha is the one modernly adopted, and the Portuguese mathematician was the first to do things this way. That he was misunderstood by his contemporaries – even three decades later – is clear from reviews of the French edition of the *Principios*, one by J. T. Mayer, in the *Göttingische gelehrte Anzeigen* of November 14, 1811, and another, quite detailed, by the Scottish mathematician J. Playfair, in the *Edinburgh Review* of November 1812. Neither author understood Anastácio da Cunha's definition of power.

Playfair writes:

"The arithmetic of powers, as it is presented in the ninth book, is one of the great peculiarities in the method of our author. This definition of power, as everyone will agree, is quite unusual; and we cannot accept that the inconveniences of following the usual method are so many as to justify such a great innovation."

And then, after defining powers with natural and positive rational exponents:

"Therefore, the idea of a power in its most general form is deduced from the simple arithmetic process of multiplication. The idea proposed by our author in place of this, although it can be shown, after many reasonings, to be the same, is infinitely more complicated to begin with."

J. T. Mayer in turn writes:

"IX. On powers and the first binomial theorem. The way in which the author approaches these subjects is new and peculiar. We doubt it will find applause. The theorem, so easy to prove, that $a^m a^n = a^{m+n}$ fills here a whole page, and it is derived from the consideration of certain infinite series. Here we must state the objection that the author totally deviates from the usual notion of power, probably due to the difficulty that fractional and negative exponents seem to present."

Anastácio da Cunha's work on powers and logarithms was praised by none other than C. F. Gauss, who, in a letter to Bessel dated November 21, 1811, writes:

"All paradoxes that some mathematicians *found* in logarithms disappear by themselves, when one does not start from the usual definition $\text{basis}^{\text{logar.}} = \text{number}$, which in truth only works when the exponent is an integral number, and does not make any sense when the exponent is imaginary – but one calls logarithm of $A$ to a magnitude such that, when one replaces $x$ by that magnitude in the series

$$1 + x + \frac{1}{2}xx + \frac{1}{6}x^3 + \text{etc.},$$

this assumes the value $A$; I see with pleasure that the Portuguese Cunha in fact chose this definition – and for that he was blamed in a bad review in our *Gelehrten Anzeigen*."



## Book 15 of the *Principios Mathematicos*

The same attitude of mind leads Anastácio da Cunha to a remarkable exposition of the bases of the Differential and Integral Calculus in Book 15 of his treatise. Again the modernity with which the subject is ordered and treated cannot fail to impress.

The author begins by distinguishing between constant and variable expressions. The crucial definition follows:

"(...) A variable which can always admit values less than any proposed quantity shall be called infinitesimal."

Next the word *function* is introduced:

"If the value of an expression $A$ depends of another expression $B$, $A$ shall be called a function of $B$ (...)"

Definition 4 presents, in the old language of fluxions, what several authors (including A. Youschkevitch and J. Mawhin) consider to be the first rigorous analytic definition of the differential. It reads as follows (where $\Gamma$ is a function):

"If we denote by $dx$ a quantity chosen to be the fluxion of $x$, we shall call the fluxion of $\Gamma x$, and denote by $d\Gamma x$, the quantity which makes $\frac{d\Gamma x}{dx}$ constant, and $\frac{\Gamma(x+dx)-\Gamma x}{dx} - \frac{d\Gamma x}{dx}$ infinitesimal or zero, if $dx$ is infinitesimal and everything that does not depend on $dx$ is constant."

Most remarkable is the proof of Proposition 1, which states:

"$x$ infinitesimal implies that $Ax + Bx^2 + Cx^3 + Dx^4 +$ &c. is infinitesimal, if the coefficients $A$, $B$, $C$, $D$, &c. are constant."

The proof reads:

"Let $n$ be the number of the coefficients $A$, $B$, $C$, $D$, &c., and let $P$ be any quantity larger than each one of them. Let Q be any proposed quantity; take $x < \frac{Q}{nP}$ and $< 1$. Then $\frac{1}{n}Q > Px$, $\frac{1}{n}Q > Px^2$, $\frac{1}{n} > Px^3$, and so on, whence $Q > Ax + Bx^2 + Cx^3 + Dx^4 +$ &c."

Anastácio da Cunha thus presents a typical 19th-century argument, reducing the treatment of infinitesimal quantities to an algebraic manipulation of inequalities involving finite and well-determined quantities.

Proposition 1 is used to prove that $dx^n = nx^{n-1}dx$ (Proposition 2).

Proposition 6 states, using our language, that the differentiability of a function implies its continuity ("If $dx$ is infinitesimal, and everything that does not depend on $x$ is constant, then $\Gamma(x+dx)-\Gamma x$ is infinitesimal."). Although Anastácio da Cunha does not make the concept of a continuous function explicit, the lucidity of this observation is remarkable.

In the remainder of Book 15 the elements of the differential calculus for functions of one or more variables are presented (including, for example, equality of mixed partial derivatives, proved assuming existence of Taylor series), as well as of the integral calculus. All the material is presented in sequential form, building on the bases set out at the beginning, with the concepts defined and used precisely and unambiguously, and with an effort to justify each statement from others coming before. The word *limit* is not used, but the modern theory of limits, and its application to the rigorous establishment of the calculus, are already essentially present.



## Other remarks about the *Principios Mathematicos*

Much more could be said about the *Principios Mathematicos*, a work maturely thought out and written with clarity and precision of language. Anastácio da Cunha makes an effort to avoid gaps in his arguments, insisting on the completely deductive character of his treatise. This leads him, in several passages of his exposition, to insert "primitive" statements in the Euclidean fashion, that is, postulates and axioms starting from which he can build his edifice. The most interesting of these statements are of course those referring to the modern parts of the treatise. And in these we see Cunha turn to "experience" to justify them. It is revealing, in light of the intellectual atmosphere in which he grew up, that he feels the need to ground the primitive assertions of his purely mathematical treatise in the same experience which is at the root of the physical sciences.

Examples of these statements are the following.

Book 8 (on positive and negative numbers, operations and various rules) ends like this: "Axiom. Experience has shown that these rules are safe."

In Book 10 (on equations), when speaking about square roots of negative numbers, Cunha writes: "... When modern Mathematicians encounter similar expressions, they do not stop their calculation; experience shows that it turns out right as long as some care is taken."

In Book 17 (dealing with the differential geometry of curves), Anastácio da Cunha feels the need to write: "Experience has shown to Geometers that any variable with infinitesimal differences between its values, when passing from positive to negative finds itself equal to 0, or to $\frac{1}{0}$)."

This last example is especially interesting. Without a construction of the real numbers and a precise notion of continuity of functions, both made explicit in the following century, this statement (today associated with the names of Bolzano and Cauchy) could never appear as a proposition, and it is remarkable to see it made explicit in this way.

## Conclusion

What is essentially new in José Anastácio da Cunha's *Principios Mathematicos* is his "foundational attitude" concerning the mathematical topics covered. This attitude is apparent in all his writings, but achieves its most interesting and original results when dealing with the controversial matters of infinite series and the calculus. It seems undeniable that in this Cunha was a remarkable forerunner of the later efforts to put infinitary processes on solid logical ground.

Francisco Gomes Teixeira, in the *História das Matemáticas em Portugal*, is accurate in his portrait. After briefly describing Anastácio da Cunha's mathematical work, he says:

> "... by this analysis we can see that its author was above all a distinguished logician.



> 18th-century mathematicians, concerned with fructifying and augmenting the great legacy bequeathed by the geometers of the previous century, developing to that purpose the great inventions made by them, did not pay attention to the logical part of their demonstrations, thereby breaking the tradition of rigorous reasoning of Greek geometers. The return to this tradition was reserved for the mathematicians of the 19th century, with the critical analysis of the doctrines of their predecessors, so as to state exactly the conditions for the application of each theorem.
>
> Anastácio da Cunha is in the 18th century one of the forerunners of the geometers who in the 19th century carried out this considerable work of logical organization of the new domains open in the world of numbers, and his writings and his name should appear in the brilliant history of this organization."

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For personal notes on Cunha by a friend, see:

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