PARTIAL SPECTRA OF SUMS OF HERMITIAN MATRICES

JOÃO FILIPE QUEIRÓ

Dedicated to Eduardo Marques de Sá, with the highest admiration

Abstract. Some remarks are made concerning the problem of describing the possible partial spectra of a sum of two Hermitian matrices with given eigenvalues.

1. Introduction

Throughout this paper, $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ denote two n-tuples of real numbers ordered so that $\alpha_1 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \cdots \geq \beta_n$.

The question to be addressed (suggested in [4]) is the following. Fix an integer $s \in \{1, \dots, n\}$, and indices $k_1, \dots, k_s$ such that $1 \leq k_1 < \cdots < k_s \leq n$. Given an s-tuple $\gamma = (\gamma_{k_1}, \dots, \gamma_{k_s})$, with $\gamma_{k_1} \geq \cdots \geq \gamma_{k_s}$, when do there exist Hermitian A and B, with spectra α and β respectively, such that γ is a part of the spectrum of $A + B$? In other words, what are the possible s-tuples $\gamma = (\gamma_{k_1}, \dots, \gamma_{k_s})$ such that, for $j = 1, \dots, s$, the j-th coordinate of γ, γ_{k_j}, is the k_j-th eigenvalue of a sum $A + B$, A and B Hermitian with spectra α and β? We will make some elementary remarks on this problem concerning particular values of s.

2. The full spectrum case

The case where $s = n$, *i.e.* we are interested in the possible (complete) spectra of sums $A + B$, A and B Hermitian with the given spectra α and β, has a long history, with connections to different parts of Mathematics, and has been solved a few years ago. We briefly recall this solution. (The interested reader can find more details in the fine surveys [2, 4].)

Denote by $E(\alpha, \beta)$ the set of possible such spectra γ. This set is easily seen to be compact and connected, as it is the image of the unitary group under a continuous mapping. It is contained in the hyperplane defined by the trace condition, which we abbreviate to $\Sigma\gamma = \Sigma\alpha + \Sigma\beta$. In [3] it was shown that it is convex, using the convexity properties of the moment mapping from symplectic geometry.

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In 1962, A. Horn conjectured that $E(\alpha, \beta)$ is completely described by a family of inequalities of the type

$$\gamma_{k_1} + \cdots + \gamma_{k_r} \leq \alpha_{i_1} + \cdots + \alpha_{i_r} + \beta_{j_1} + \cdots + \beta_{j_r}$$

where $r \in \{1, ..., n\}$ and $i_1 < \ldots < i_r, j_1 < \ldots < j_r, k_1 < \ldots < k_r$, or, in short,

$$\Sigma\, \gamma_K \;\leq\; \Sigma\, \alpha_I \;+\; \Sigma\, \beta_J$$

where $I = (i_1, \ldots, i_r), J = (j_1, \ldots, j_r), K = (k_1, \ldots, k_r)$. A consequence of this would be that $E(\alpha, \beta)$ is a convex polytope.

The question is to identify the right triples (I, J, K). Horn makes an elaborate conjecture on this, which, in sightly changed form, reads as follows:

Write $\lambda(I) = (i_r - r, \ldots, i_2 - 2, i_1 - 1)$ and similarly for $\lambda(J)$ and $\lambda(K)$. Then Horn's conjecture is that $\gamma \in E(\alpha, \beta)$ if and only if

$$\Sigma\gamma = \Sigma\alpha + \Sigma\beta$$

and

$$\Sigma\gamma_K \leq \Sigma\alpha_I + \Sigma\beta_J \text{ whenever } \lambda(K) \in E[\lambda(I), \lambda(J)] \quad (\text{for all } r, 1 \leq r < n).$$

This means that the set $E(\alpha, \beta)$ is described recursively from lower dimensions.

We present below the list of Horn inequalities for $n = 2$ and $n = 3$ (apart from the trace equalities).

$n = 2$:

$$\gamma_1 \leq \alpha_1 + \beta_1 \qquad \gamma_2 \leq \alpha_1 + \beta_2 \qquad \gamma_2 \leq \alpha_2 + \beta_1$$

$n = 3$:

$$\begin{array}{ll}
\gamma_1 \leq \alpha_1 + \beta_1 & \gamma_1 + \gamma_2 \leq \alpha_1 + \alpha_2 + \beta_1 + \beta_2 \\
\gamma_2 \leq \alpha_1 + \beta_2 & \gamma_1 + \gamma_3 \leq \alpha_1 + \alpha_2 + \beta_1 + \beta_3 \\
\gamma_3 \leq \alpha_1 + \beta_3 & \gamma_2 + \gamma_3 \leq \alpha_1 + \alpha_2 + \beta_2 + \beta_3 \\
\gamma_2 \leq \alpha_2 + \beta_1 & \gamma_1 + \gamma_3 \leq \alpha_1 + \alpha_3 + \beta_1 + \beta_2 \\
\gamma_3 \leq \alpha_2 + \beta_2 & \gamma_2 + \gamma_3 \leq \alpha_1 + \alpha_3 + \beta_1 + \beta_3 \\
\gamma_3 \leq \alpha_3 + \beta_1 & \gamma_2 + \gamma_3 \leq \alpha_2 + \alpha_3 + \beta_1 + \beta_2
\end{array}$$

The conjecture was proved in the late 1990s, as a result of work of Klyachko in [7] and Knutson and Tao in [9]. This work involves, among other subjects, the intersection of Schubert varieties as well the representations of the general linear group, including the combinatorics of tableaux.

Other references on the problem are [3, 8, 12, 13].

The number of inequalities in Horn's list grows very rapidly with n. In [1] and [10], the question of the independence of these inequalities for each n was studied.

3. The case $s = 1$

Here we are interested, given k, in the possible numbers γ_k occurring as the k-th eigenvalue of sums $A + B$, A and B Hermitian with the given spectra α and β.

The following inequalities concerning a single eigenvalue of $A + B$ have been known for a long time, and associated with the name of Weyl:

$$\gamma_{i+j-1} \leq \alpha_i + \beta_j \quad , \quad \gamma_{i+j-n} \geq \alpha_i + \beta_j$$

(for all admissible values of the indices). The first is easily proved from the extremal characterizations of eigenvalues of Hermitian matrices using the associated quadratic forms, and the second follows from the first applied to $-A$ and $-B$.

We rewrite Weyl's inequalities as follows:

$$\alpha_i + \beta_{k-i+n} \;\leq\; \gamma_k \quad (i = k, \dots, n) \qquad \text{and} \qquad \gamma_k \;\leq\; \alpha_i + \beta_{k-i+1} \quad (i = 1, \dots, k)$$

or

$$\max_{k \leq i \leq n} \; \alpha_i + \beta_{k-i+n} \;\leq\; \gamma_k \;\leq\; \min_{1 \leq i \leq k} \; \alpha_i + \beta_{k-i+1} \, .$$

Theorem 3.1 ([6]; see also [11, 14])**.** *These conditions are sufficient.*

Proof. The set of possible γ is an interval. By the necessity, that interval is contained in

$$[\; \max_{k \leq i \leq n} \; \alpha_i + \beta_{k-i+n} \;,\; \min_{1 \leq i \leq k} \; \alpha_i + \beta_{k-i+1} \,] \, .$$

The lower bound is attained by the k-th eigenvalue of $A + B$, where

$$A = \operatorname{diag}(\alpha_1, \dots, \alpha_n) \;\; , \;\; B = \operatorname{diag}(\beta_k, \beta_{k-1}, \dots, \beta_2, \beta_1, \beta_{k+1}, \beta_{k+2}, \dots, \beta_n) \, .$$

The upper bound is attained by the k-th eigenvalue of $A + B$, where

$$A = \operatorname{diag}(\alpha_1, \dots, \alpha_n) \;\; , \;\; B = \operatorname{diag}(\beta_1, \beta_2, \dots, \beta_{k-1}, \beta_n, \beta_{n-1}, \dots, \beta_{k+1}, \beta_k) \, . \;\; \square$$

Two comments are in order concerning this result. First, the extreme points of the realizable set are produced with diagonal matrices. Second, the relevant inequalities are precisely those appearing in Horn's list as explicit bounds on a single eigenvalue: the first family of Weyl inequalities, $\gamma_{i+j-1} \leq \alpha_i + \beta_j$, is clearly the list of 1-term inequalities appearing there; the second family, $\gamma_{i+j-n} \geq \alpha_i + \beta_j$, consists exactly of the inequalities obtained from the trace condition together with the $(n-1)$-term inequalities in the Horn list [4], since if I has length $n-1$, the $(n-1)$-tuple $\lambda(I)$ has only 1's and 0's, and in that situation it is not difficult to find all possible cases in which $\lambda(K) \in E[\lambda(I), \lambda(J)]$.

4. Note on other values of s

In light of the last comment in the previous section, a natural conjecture would be that, for any s, the restrictions involving $\gamma = (\gamma_{k_1}, \ldots, \gamma_{k_s})$ in our problem would be precisely those appearing in Horn's list as explicit bounds on sums of entries in γ (both upper and lower bounds, the latter obtained using the trace condition).

Again as remarked in [4], this conjecture is not true. The example given there is $n = 3$, $s = 2$, $\gamma = (\gamma_1, \gamma_3)$. The explicit bounds from Horn's list in this case are:

$$\alpha_1 + \beta_3 \,,\, \alpha_2 + \beta_2 \,,\, \alpha_3 + \beta_1 \;\leq\; \gamma_1 \;\leq\; \alpha_1 + \beta_1$$

$$\alpha_3 + \beta_3 \;\leq\; \gamma_3 \;\leq\; \alpha_1 + \beta_3 \,,\, \alpha_2 + \beta_2 \,,\, \alpha_3 + \beta_1$$

$$\alpha_1+\alpha_3+\beta_2+\beta_3 \,,\, \alpha_2+\alpha_3+\beta_1+\beta_3 \;\leq\; \gamma_1+\gamma_3 \;\leq\; \alpha_1+\alpha_2+\beta_1+\beta_3 \,,\, \alpha_1+\alpha_3+\beta_1+\beta_2$$

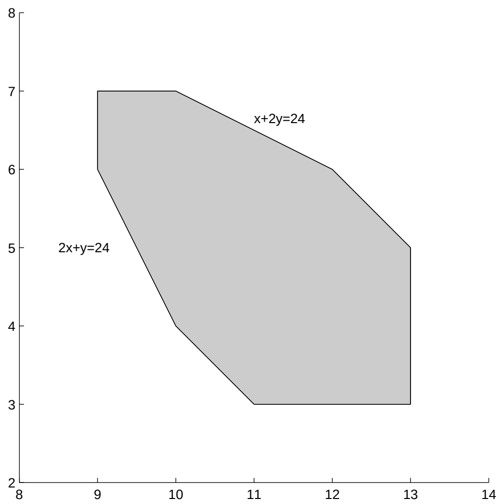
But these conditions are not sufficient, as they do not force the second eigenvalue of the sum (which must be equal to $\alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 + \beta_3 - \gamma_1 - \gamma_3$) to be between γ_1 and γ_3.

It turns out that the introduction of the (obviously necessary) inequalities needed to solve that ordering problem,

$$\gamma_1 + 2\gamma_3 \;\leq\; \alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 + \beta_3 \;\leq\; 2\gamma_1 + \gamma_3 \,,$$

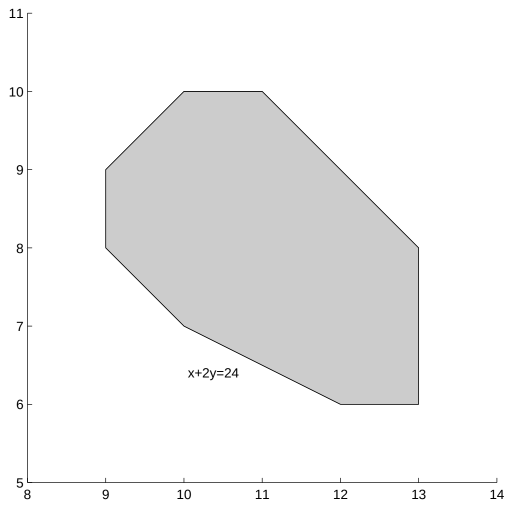
yields the complete answer in this case.

The following illustrates the polygon of realizable γ when the given spectra are $\alpha = (6, 4, 2), \beta = (7, 4, 1)$:



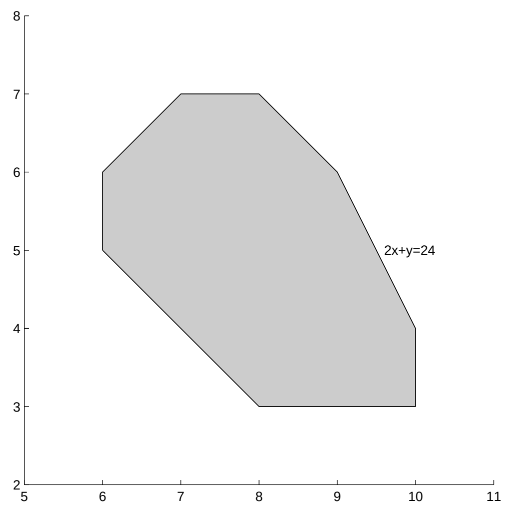
If $\gamma = (\gamma_1, \gamma_2)$, there is only additional condition:

$$\gamma_1 + 2\gamma_2 \ \geq \ \alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 + \beta_3 \,.$$



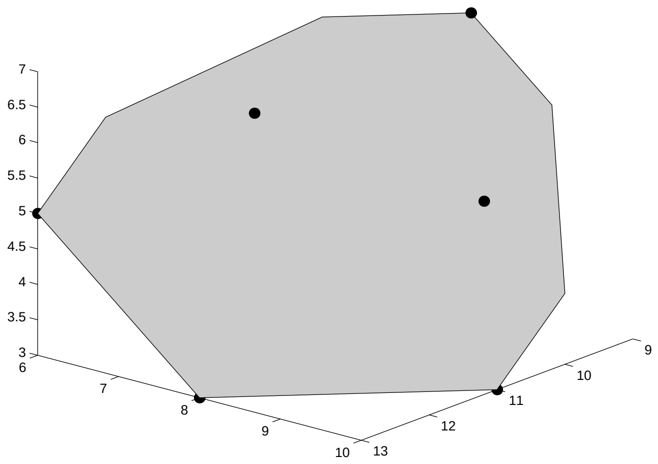
If $\gamma = (\gamma_2, \gamma_3)$, the additional condition is

$$2\gamma_2 + \gamma_3 \ \leq \ \alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 + \beta_3 \,.$$



These are particular instances of the case $s = n - 1$, to which we dedicate the next section.

Another comment is that in general we cannot expect the vertices of the realizable set to be produced with diagonal matrices. This is already the case in the $n = 3$ example above. The next illustration shows the polygon $E(\alpha, \beta)$ in $\mathbb{R}^3$ for the same triples α and β:



We have marked with a black dot the six points in $E(\alpha, \beta)$ produced with diagonal A and B. There are four vertices not obtained with diagonal matrices: $(10, 10, 4), (9, 9, 6), (10, 7, 7), (12, 6, 6)$.

That is why in general the partial spectrum problem does not follow trivially from the full spectrum case: we wish to describe the projection of $E(\alpha, \beta)$ onto the coordinate s-plane spanned by the $k_1, \ldots, k_s$ canonical vectors, but we don't know the vertices of $E(\alpha, \beta)$.

5. The case $s = n - 1$

Here we are interested in the possible $(n-1)$-tuples

$$\gamma = (\gamma_1, \ldots, \gamma_{h-1}, \gamma_{h+1}, \ldots, \gamma_n)$$

occurring as part of the spectra of sums $A + B$, A and B Hermitian with the given spectra α and β.

This is the simplest case of all, because the missing eigenvalue is determined from the trace condition.

Theorem 5.1. *The complete restrictions in this case are those appearing in Horn's list as explicit bounds on sums of entries in γ (both upper and lower bounds, the latter obtained using the trace condition), together with*

$$\gamma_{h+1} + \sum_{k \neq h} \gamma_k \ \leq \ \sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{n} \beta_j \ \leq \ \gamma_{h-1} + \sum_{k \neq h} \gamma_k \tag{5.1}$$

where the first inequality is not present if $h = n$, and the second inequality is not present if $h = 1$.

Proof. The necessity is obvious. For the sufficiency, we have to put

$$\gamma_h = \sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{n} \beta_j - \sum_{k \neq h} \gamma_k \, .$$

From (5.1) it follows that $\gamma_{h-1} \geq \gamma_h \geq \gamma_{h+1}$. All $\leq$ inequalities in Horn's list involving only the entries in γ (not involving γ_h) are present in the hypothesis. The conditions in Horn's list involving γ_h are precisely those coming from the $\geq$ inequalities in Horn's list involving only the entries in γ (together with the trace condition), and therefore also follow directly from the hypothesis. □

The statements for $n = 3$ and $s = 2$ in the previous section are direct applications of this theorem.

6. The case $s = n - 2$

The next natural case is $s = n - 2$. We end this article with just a few remarks about it, starting with a special situation.

Theorem 6.1. *Let $n = 4$, $\gamma = (\gamma_2, \gamma_3)$, with $\gamma_2 \geq \gamma_3$. There exist 4×4 Hermitian A and B, with spectra α and β respectively, such that γ_2 and γ_3 are the second and the third eigenvalues of $A + B$ if and only if the entries in γ satisfy the inequalities in Horn's list involving only γ_2, γ_3 and $\gamma_2 + \gamma_3$ (both upper and lower bounds, the latter obtained using the trace condition), together with*

$$2\gamma_2 + \gamma_3 \ \leq \ \alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 + \beta_3 \, , \tag{6.1}$$

$$\gamma_2 + 2\gamma_3 \ \geq \ \alpha_2 + \alpha_3 + \alpha_4 + \beta_2 + \beta_3 + \beta_4 \, . \tag{6.2}$$

Proof. The necessity is obvious, with (6.1) coming from

$$\gamma_1 + \gamma_2 + \gamma_3 \ \leq \ \alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 + \beta_3$$

and (6.2) coming from

$$\gamma_1 \le \alpha_1 + \beta_1$$

both in Horn's list.

Now the sufficiency. We have to exhibit γ_1 and γ_4 such that $\gamma_1 \ge \gamma_2 \ge \gamma_3 \ge \gamma_4$ and these numbers satisfy Horn's inequalities with the 4-tuples $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $(\beta_1, \beta_2, \beta_3, \beta_4)$.

For the record, the inequalities in Horn's list involving only γ_2, γ_3 and $\gamma_2 + \gamma_3$ are the following:

$$\alpha_2 + \beta_4 \,,\, \alpha_3 + \beta_3 \,,\, \alpha_4 + \beta_2 \le \gamma_2 \le \alpha_1 + \beta_2 \,,\, \alpha_2 + \beta_1$$

$$\alpha_3 + \beta_4 \,,\, \alpha_4 + \beta_3 \le \gamma_3 \le \alpha_1 + \beta_3 \,,\, \alpha_2 + \beta_2 \,,\, \alpha_3 + \beta_1$$

$$\left.\begin{matrix} \alpha_2 + \alpha_3 + \beta_3 + \beta_4 \\ \alpha_2 + \alpha_4 + \beta_2 + \beta_4 \\ \alpha_3 + \alpha_4 + \beta_2 + \beta_3 \end{matrix}\right\} \le \gamma_2 + \gamma_3 \le \left\{\begin{matrix} \alpha_1 + \alpha_2 + \beta_2 + \beta_3 \\ \alpha_1 + \alpha_3 + \beta_1 + \beta_3 \\ \alpha_2 + \alpha_3 + \beta_1 + \beta_2 \end{matrix}\right.$$

We first define γ_1 as the minimum of the following five numbers:

$$\alpha_1 + \beta_1$$
$$\alpha_1 + \alpha_2 + \beta_1 + \beta_2 - \gamma_2$$
$$\alpha_1 + \alpha_2 + \beta_1 + \beta_3 - \gamma_3$$
$$\alpha_1 + \alpha_3 + \beta_1 + \beta_2 - \gamma_3$$
$$\alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 + \beta_3 - \gamma_2 - \gamma_3$$

This ensures two things: that $\gamma_1 \ge \gamma_2$, as γ_2 is easily seen, using the hypothesis, to be less than or equal to all five numbers, and that γ_1 satisfies all inequalities in Horn's list that give upper bounds for it or for sums of it with the given γ_2 and γ_3.

Next, of course, we put

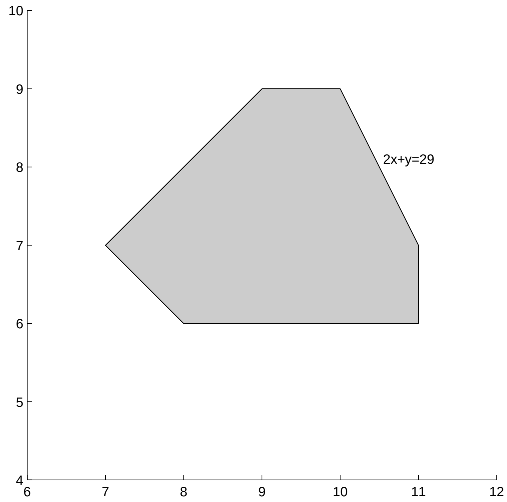
$$\gamma_4 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \beta_1 + \beta_2 + \beta_3 + \beta_4 - \gamma_1 - \gamma_2 - \gamma_3 \,.$$

The rest of the proof consists of a tedious checking that $\gamma_4 \le \gamma_3$ and that γ_4 satisfies all remaining relevant conditions in Horn's list: upper bounds for γ_4, $\gamma_2 + \gamma_4$, $\gamma_3 + \gamma_4$, and $\gamma_2 + \gamma_3 + \gamma_4$.

The others (upper bounds for $\gamma_1 + \gamma_4$, $\gamma_1 + \gamma_2 + \gamma_4$, and $\gamma_1 + \gamma_3 + \gamma_4$) follow directly from the hypothesis (lower bounds for $\gamma_2 + \gamma_3$, γ_3, and γ_2, respectively).

It is worth remarking, although not unexpected, that (6.1) is only used in proving that $\gamma_1 \ge \gamma_2$, and (6.2) is only used in proving that $\gamma_4 \le \gamma_3$. □

The following illustrates the polygon of realizable γ when the given spectra are $\alpha = (6, 4, 3, 2)$, $\beta = (7, 5, 4, 1)$. We have marked the side coming from inequality (6.1). Condition (6.2) does not restrict the set in this example.



Conditions (6.1) and (6.2) can be found by scanning Horn's list for inequalities that, together with the ordering of the γ's, imply extra restrictions involving the partial spectrum $\gamma = (\gamma_2, \gamma_3)$. (That they are exactly what is needed requires a separate proof.)

This suggests a way to identify the extra conditions in other cases.

For example, again for $n = 4$, if $\gamma = (\gamma_1, \gamma_2)$ we should add to the explicit upper and lower bounds for γ_1, γ_2 and $\gamma_1 + \gamma_2$ coming from Horn's list the following conditions:

$$\gamma_1 + 2\gamma_2 \ \geq \ \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 + \beta_2 + \beta_3 + \beta_4 \\ \alpha_1 + \alpha_2 + \alpha_4 + \beta_1 + \beta_3 + \beta_4 \\ \alpha_1 + \alpha_3 + \alpha_4 + \beta_1 + \beta_2 + \beta_4 \\ \alpha_2 + \alpha_3 + \alpha_4 + \beta_1 + \beta_2 + \beta_3 \end{cases}$$

$$\gamma_1 + 3\gamma_2 \ \geq \ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \beta_1 + \beta_2 + \beta_3 + \beta_4$$

For $\gamma = (\gamma_1, \gamma_3)$ the extra conditions are:

$$\left.\begin{matrix} \alpha_1 + \alpha_3 + \alpha_4 + \beta_2 + \beta_3 + \beta_4 \\ \alpha_2 + \alpha_3 + \alpha_4 + \beta_1 + \beta_3 + \beta_4 \end{matrix}\right\} \ \leq \ \gamma_1 + 2\gamma_3 \ \leq \ \alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 + \beta_3$$

$$2\gamma_1 + \gamma_3 \ \geq \ \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 + \beta_2 + \beta_3 + \beta_4 \\ \alpha_1 + \alpha_2 + \alpha_4 + \beta_1 + \beta_3 + \beta_4 \\ \alpha_1 + \alpha_3 + \alpha_4 + \beta_1 + \beta_2 + \beta_4 \\ \alpha_2 + \alpha_3 + \alpha_4 + \beta_1 + \beta_2 + \beta_3 \end{cases}$$

And so on. This gives an idea of how to find the extra restrictions for all values of n and s. It is not clear to me at the moment how to go about proving the sufficiency part in the resulting statements. It seems natural to try to take advantage of the recursive description of Horn's inequality list.

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Departamento de Matemática
Universidade de Coimbra
3001-454 Coimbra, Portugal
E-mail address: jfqueiro@mat.uc.pt