s-NUMBERS OF MATRICES AND THE SEPARATION THEOREM

by JOÃO FILIPE QUEIRÓ[32] LAA 121 (1989), 653–658

Introduction

Throughout, $\mathbf{F}$ will denote either the field $\mathbf{R}$ of real numbers or the field $\mathbf{C}$ of complex numbers. Given $A \in \mathbf{F}^{m,n}$, its singular values are denoted by $\sigma_1(A) \geqslant \sigma_2(A) \geqslant \cdots \geqslant \sigma_{\min(m,n)}(A)$.

The well-known separation (or interlacing) theorem for singular values [7] states that if $B \in \mathbf{F}^{m-p,n-q}$ is s submatrix of $A \in \mathbf{F}^{m,n}$, then

$$\sigma_k(A) \geqslant \sigma_k(B), \qquad k = 1,2,\dots,\min\{m-p, n-q\},$$

$$\sigma_k(B) \geqslant \sigma_{k+p+q}(A), \qquad k = 1,2,\dots,\min\{m-p-q, n-p-q\}$$

(the *interlacing inequalities*). Reciprocally, given $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_{\min(m,n)} \geqslant 0$ and $\beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant \beta_{\min(m-p,n-q)} \geqslant 0$, if

$$\alpha_k \geqslant \beta_k, \qquad k = 1,2,\dots,\min\{m-p, n-q\},$$

$$\beta_k \geqslant \alpha_{k+p+q}, \qquad k = 1,2,\dots,\min\{m-p-q, n-p-q\},$$

then there exists $A \in \mathbf{F}^{m,n}$ with singular values $\{\alpha_i\}$ and containing a submatrix $B \in \mathbf{F}^{m-p,n-q}$ with singular values $\{\beta_i\}$ [7]. Equivalently, any $(m-p)\times(n-q)$ matrix with singular values $\{\beta_i\}$ can be augmented to a $m \times n$ matrix with singular values $\{\alpha_i\}$.

The problem that interests us is: How much of this remains true for various generalizations of the singular values?

The singular values are intimately connected with the Euclidean norm in the spaces $\mathbf{F}^n$, $\chi(x) := (\sum_j |x_j|^2)^{1/2}$. The generalizations involve taking other norms.

We need some notation. ψ, φ, ν will denote norms (or families of norms) in $\bigcup_n \mathbf{F}^n$, such as the Hölder norms $\chi_t(x) := (\sum_j |x_j|^t)^{1/t}$, $1 \leqslant t \leqslant \infty$ (note that $\chi_2 = \chi$). Analogously, μ will denote a norm in $\bigcup_{m,n} \mathbf{F}^{m,n}$. $S_{\psi\varphi}(A)$ is the norm of $A \in \mathbf{F}^{m,n}$ as an operator between the normed spaces $(\mathbf{F}^n, \varphi)$ and $(\mathbf{F}^m, \psi)$, i.e. $S_{\psi\varphi}(A) = \sup_{x \neq 0} \psi(Ax)/\varphi(x)$. If $A_{\cdot 1}, \dots, A_{\cdot n}$ are the columns of

[32] Departamento Matemática, Universidade Coimbra, 3000 Coimbra, Portugal. Supported by INIC.

$A \in \mathbf{F}^{m,n}$ and if φ is absolute, we define $[A]_{\varphi\psi} = \varphi(\psi(A_{\cdot 1}), \dots, \psi(A_{\cdot n}))$ (these are the composite norms of Maitre [3]).

The following definition was given by Pietsch for operators between Banach spaces.

DEFINITION [4]. A mapping $s: \bigcup_{m,n} \mathbf{F}^{m,n} \to \mathbf{R}^{\mathbf{N}}$ is an s-number function if, for each pair of norms ψ, φ, it associates with every matrix A a sequence $(s_k^{\psi\varphi}(A))_{k \in \mathbf{N}}$ such that

(1) $S_{\psi\varphi}(A) = s_1^{\psi\varphi}(A) \geqslant s_2^{\psi\varphi}(A) \geqslant \cdots \geqslant 0$,
(2) $s_k^{\psi\varphi}(A+B) \leqslant s_k^{\psi\varphi}(A) + S_{\psi\varphi}(B)$,
(3) $s_k^{\nu\nu'}(BAC) \leqslant S_{\nu\psi}(B) s_k^{\psi\varphi}(B) S_{\varphi\nu'}(C)$,
(4) $\operatorname{rank}(A) < k \Rightarrow s_k^{\psi\varphi}(A) = 0$,
(5) $k \leqslant n \Rightarrow s_k^{\psi\psi}(I_n) = 1$

(for any $A, B, C, \psi, \varphi, \nu, \nu', k$).

EXAMPLES

The approximation numbers:

$$a_k^{\psi\varphi}(A) := \inf\{ S_{\psi\varphi}(A - X) : \operatorname{rank}(X) < k \}.$$

The Gelfand numbers:

$$g_k^{\psi\varphi}(A) := \inf_{\dim E = n-k+1} \sup_{0 \neq x \in E} \frac{\psi(Ax)}{\varphi(x)}.$$

The Bernstein numbers:

$$b_k^{\psi\varphi}(A) := \sup_{\dim E = k} \inf_{0 \neq x \in E} \frac{\psi(Ax)}{\varphi(x)}.$$

The Kolmogorov numbers:

$$h_k^{\psi\varphi}(A) := g_k^{\varphi^d\psi^d}(A^*) = \inf_{\dim E = k-1} \sup_{\varphi(x) \leqslant 1} \inf_{y \in E} \psi(Ax - y).$$

The Mitiagin numbers:

$$m_k^{\psi\varphi}(A) := b_k^{\varphi^d\psi^d}(A^*).$$

SOME PROPERTIES.

(1) *For any s, the mapping* $(\psi, \varphi, A) \to s_k^{\psi\varphi}(A)_{k\in\mathbf{N}}$ *is continuous.*

(2) *If* A *is* $n \times n$, *then for any* s, $s_n^{\psi\varphi}(A) = \inf_{x\neq 0}[\psi(Ax)/\varphi(x)]$.

(3) *If* $A = \mathrm{diag}(\alpha_1,\dots,\alpha_n)$ *and* ψ *is absolute, then for any* s, $s_k^{\psi\psi}(A) = |\alpha_{\tau(k)}|$, $k = 1,\dots,n$ (*where* τ *is such that* $|\alpha_{\tau(1)}| \geqslant \cdots \geqslant |\alpha_{\tau(n)}|$).

THEOREM [4]. *If s is any s-number function, then* $s_k^{\chi\chi}(\cdot) = \sigma_k(\cdot)$ *for all* k.

THE SEPARATION THEOREM.

(i) *Let s be any s-number function, B a submatrix of A. If* ψ *is absolute and* φ *is arbitrary, then* $s_k^{\psi\varphi}(A) \geqslant s_k^{\psi\varphi}(B)$ *for all k.*

(ii) *Let s be an additive s-number function,* $B \in \mathbf{F}^{m-p,n-q}$ *a submatrix of* $A \in \mathbf{F}^{m,n}$. *If* φ *is absolute and* ψ *is arbitrary, then* $s_k^{\psi\varphi}(B) \geqslant s_{k+p+q}^{\psi\varphi}(A)$ *for all k.*

REMARKS.

(1) Absolute ($\Leftrightarrow$ monotone) can be relaxed to orthant-monotonic (see [2] for the definition in the real case).

(2) s additive [4] means $s_{k+j-1}^{\psi\varphi}(A+B) \leqslant s_k^{\psi\varphi}(A) + s_j^{\psi\varphi}(B)$. Examples: a_k, g_k, h_k. This condition can be relaxed to a weaker one, satisfied, for example, by the b_k and the m_k (it is not known whether they are additive).

Generalized Approximation Numbers

Another possible generalization is the following: As mentioned before, $\sigma_k(A) = a_k^{\chi\chi}(A) = \inf\{S_{\chi\chi}(A - X) : \mathrm{rank}(X) < k\}$. Now take μ as any norm (= family of norms) in $\bigcup_{m,n}\mathbf{F}^{m,n}$, and define

$$a_k^{\mu}(A) := \inf\{\mu(A - X) : \mathrm{rank}(X) < k\}.$$

Then [5]

(i) $a_k^\mu(B) \geqslant a_{k+p+q}^\mu(A)$ if $B \in \mathbf{F}^{m-p,n-q}$ is a submatrix of $A \in \mathbf{F}^{m,n}$;
(ii) $a_k^\mu(A) \geqslant a_k^\mu(B)$ if μ(matrix) is always $\geqslant \mu$(submatrix).

The Reciprocal of Interlacing

Given the β's denote by Δ the subset of $\mathbf{R}^{\min(m,n)}$ whose elements are the sequences of α's which interlace with the β's. Our strategy to find examples in which the reciprocal of interlacing fails is to choose the β's so that Δ will have some points with all coordinates equal, and see what happens.

Consider first the μ-approximation numbers just defined. For example, if μ is unitarily invariant, the situation is radical:

THEOREM. *Let μ be unitarily invariant. If A is square $n \times n$ and $\neq 0$, and if $a_1^\mu(A) = a_n^\mu(A)$, then $\mu = S_{XX}$ $(= \sigma_1)$.*

Another example which shows that the condition is highly restrictive may be obtained considering Maitre's norms:

THEOREM. *Let $\mu = [\cdot]_{\varphi\psi}$ (φ absolute). The nonzero matrix $A \in \mathbf{F}^{n,n}$ has all its μ-approximation numbers equal, say to c, if and only if there exists $D = \operatorname{diag}(d_1, \dots, d_n) \in \mathbf{F}^{n,n}$ nonsingular such that $\varphi(x) = \chi_\infty(Dx)$ and $\psi(x) = c\chi_1(D^{-1}A^{-1}x)$.*

EXAMPLES WITH s-NUMBER FUNCTIONS. Let $\psi = \varphi$. Let s be an s-number function. Let $m = n$, $p = q$. Let $p \geqslant n/2$. Choose $\beta_1 = \cdots = \beta_{n-p} = \beta$ $(\geqslant 0)$. Then

$$\Delta = \{(\alpha_1, \dots, \alpha_n) : \alpha_1 \geqslant \cdots \geqslant \alpha_n \geqslant 0;\ \alpha_k \geqslant \beta,\ k = 1, \dots, n-p\}.$$

Note that Δ contains the half line $\{(\xi, \dots, \xi) : \xi \geqslant \beta\}$.

Suppose that, for every $(\alpha_1, \dots, \alpha_n) \in \Delta$, there exists A $n \times n$ with ψ, ψ-s-numbers $\alpha_1, \dots, \alpha_n$ and containing an $(n-p)\times(n-p)$ submatrix with ψ, ψ-s-numbers $\beta, \dots, \beta$. For the points on the half line, this would mean $\sup_{x \neq 0}[\psi(Ax)/\psi(x)] = \inf_{x \neq 0}[\psi(Ax)/\psi(x)] = \xi$, and A/ξ would be an isometry of ψ. Hence for all $\theta \in [-1, 1]$ there would exist an isometry of ψ containing an $(n-p)\times(n-p)$ submatrix with all ψ, ψ-s-numbers equal to θ. This is a very strong condition on ψ. For example, it already leaves out the

Hölder norms, since, for $1 \leqslant t \leqslant \infty$, $t \neq 2$, the isometries of χ_t are just the "generalized permutations" (modulus-1 elements instead of 1's only) [1].

QUESTION. If ψ is a symmetric norm, does the condition imply that $\psi = \chi$? (For $n = 2$ the answer is yes.)

QUESTION. If interlacing is not enough, what other conditions are there?

An example, among a few others, where the answer can be found exactly is the following: Let $m = n = 2$, $p = q = 1$, $\psi = \chi_1$. When x, y, z vary in $\mathbf{F}$ ($\mathbf{R}$ or $\mathbf{C}$), the χ_1, χ_1-s-numbers of the 2×2 matrix

$$\begin{bmatrix} \beta & x \\ z & y \end{bmatrix}$$

describe the following region:

$$([\beta, +\infty) \times [0, \beta]) \cup \left\{ (u, v) : u \geqslant \beta, 0 \leqslant v \leqslant \frac{\beta^2 + (u - \beta)^2}{u} \right\}.$$

(Note the nonconvexity.)

A "POSITIVE" EXAMPLE. Interestingly enough, there are cases in which the interlacing inequalities are the only relations between the s-numbers of the matrices and submatrices involved.

Let $m = n + 1$, $p = q = n$ ($n = m + 1$, $p = q = m$ would also do). Let ψ be absolute. Let s be an s-number function for which the separation theorem holds (see above). Then, if $\alpha_1 \geqslant \beta_1 \geqslant \geqslant \alpha_2 \geqslant \beta_2 \geqslant \cdots \geqslant \alpha_n \geqslant \beta_n \geqslant 0$, it is possible to show that there exist $x_1, x_2, \ldots, x_n$ such that the $(n+1) \times n$ matrix

$$\begin{bmatrix} \beta_1 & & & \\ & \beta_2 & & \bigcirc \\ & & \ddots & \\ & \bigcirc & & \beta_n \\ x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

has ψ, ψ-s-numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$. For $\psi = \chi$, i.e. for the ordinary singular values, see [6] (in fact, in that case this particular situation is enough to yield the full converse of interlacing). For arbitrary ψ, a homotopy argument is used to reduce the question to the ordinary (Euclidean) case. The only problem (as the reader of [6] will notice) is the fulfillment of a technical condition involving the s-numbers of a direct sum. For $\psi = \chi$ the condition is automatic. For general ψ its study leads to a new set of problems.

Details will appear elsewhere.

REFERENCES

1 S. Banach, *Théorie des Opérations Linéaires*, Monogr. Mat., Warsaw, 1932.
2 D. Gries, Characterizations of certain classes of norms, *Numer. Math.* 10:30–41 (1967).
3 J.-F. Maitre, Norme composée et norme associée généralisée d'une matrice, *Numer. Math.* 10:132–141 (1967).
4 A. Pietsch, s-Numbers of operators in Banach spaces, *Studia Math.* 51:201–223 (1974).
5 J. F. Queiró, Generalized approximation numbers, *Portugal. Math.* 43:157–166 (1985/1986).
6 J. F. Queiró, On the interlacing property for singular values and eigenvalues, *Linear Algebra Appl.* 97:23–28 (1987).
7 R. C. Thompson, Principal submatrices IX: Interlacing inequalities for singular values of submatrices, *Linear Algebra Appl.* 5:1–12 (1972).

OBSERVABILITY OF LINEAR POSITIVE DYNAMIC SYSTEMS

by M. D. SOTO TORRES and R. FERNANDEZ LECHON[33]

1. *Introduction*

The input and output structure of a system can significantly influence the available means for control. Two fundamental concepts characterizing the dynamic implications of input and output structure are the dual concepts of controllability and observability. Controllability concerns the possibility of steering the state from the input, while observability analyzes the possibility of estimating the state from the output.

Various authors [for example, Kalman (1960, 1963), Silverman (1971)] have found necessary and sufficient conditions for controllability and observability of dynamic systems without constraints, that is, neither the variables nor the parameters of the system have to satisfy conditions. Also, various

[33]Departamento de Economía Aplicada, Facultad de Ciencias Económicas y Empresariales, Universidad de Valladolid, Avda, Valle Esgueva 6, 47011 Valladolid, Spain