THEOREM 3.2. *Assume that the Hamiltonian matrix H has no eigenvectors of the form* $(0^T, y^T)^T$ *with associated eigenvalue* λ, $\operatorname{Re}\lambda = 0$. *Then*:

(i) *A strong solution exists if and only if the Hamiltonian matrix H has no eigenvectors of the form* $(0^T, y^T)^T$ *with associated eigenvalue* λ, $\operatorname{Re}\lambda < 0$ (*(A, B) is stabilizable*).

(ii) *The strong solution is unique, maximal, and nonnegative definite.*

(iii) *The strong solution is stabilizing (or positive definite) if and only if H has no eigenvalues with real part equal to zero.*

(iv) *The strong solution is the unique nonnegative definite solution of the ARE if and only if H has no eigenvectors of the form* $(x^T, 0^T)^T$ *with associated eigenvalue* λ, $\operatorname{Re}\lambda > 0$ (*(A, C) is detectable*).

THEOREM 3.3 [4, 2]. *The strong solution of the ARE exists and is unique if and only if H has no eigenvectors of the form* $(0^T, y^T)^T$ *with associated eigenvalue* λ, $\operatorname{Re}\lambda \leqslant 0$ (*(A, B) is stabilizable*).

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SOME RESULTS AND PROBLEMS ON s-NUMBERS

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Introduction

The study of s-numbers (generalized singular values) leads to numerous interesting problems. In general, these arise when we try to extend to s-numbers results which are known and well understood for ordinary singular values.

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This talk is meant as a quick survey of a few such problems. We shall work with matrices over **F** (= **R** or **C**). As in [7], ψ, φ, ν will denote norms, or families of norms, on the columns over **F**, such as the Hölder norms $\chi_p := (\Sigma_j |x_j|^p)^{1/p}$, $1 \leq p \leq \infty$, and $S_{\psi\varphi}(A)$ is the operator norm $\sup_{x \neq 0} \psi(Ax)/\varphi(x)$.

s-Number Functions

If in several classical characterizations of singular values we replace the usual Euclidean norm $\chi := \chi_2$ with other norms, we obtain families of numbers associated with matrices (or operators) which are generically called *s-number functions*. In [4], A. Pietsch has given an axiomatic definition of these functions for operators between Banach spaces. I refer the reader to [7], where the finite-dimensional matrix version of this axiomatic definition is presented.

Examples of s-number functions are the *approximation numbers* $a_k^{\psi\varphi}(A) := \inf\{S_{\psi\varphi}(A - X): \operatorname{rank}(X) < k\}$, the *Gelfand numbers*

$$g_k^{\psi\varphi}(A) := \inf_{\dim E = n-k+1} \sup_{0 \neq x \in E} \frac{\psi(Ax)}{\varphi(x)},$$

and the *Bernstein numbers*

$$b_k^{\psi\varphi}(A) := \sup_{\dim E = k} \inf_{0 \neq x \in E} \frac{\psi(Ax)}{\varphi(x)}.$$

Using the concept of dual norm we can define the *Kolmogorov numbers* $h_k^{\psi\varphi}(A) := g_k^{\varphi^d\psi^d}(A^*)$ and the *Mitiagin numbers* $m_k^{\psi\varphi}(A) := b_k^{\varphi^d\psi^d}(A^*)$.

General properties of s-number functions include:

(1) *For any s and all k, the mapping $(\psi,\varphi,A) \to s_k^{\psi\varphi}(A)$ is continuous.*

(2) *If A is $n \times n$, then for any s, $s_n^{\psi\varphi}(A) = \inf_{x \neq 0} \psi(Ax)/\varphi(x)$.*

(3) *If P and Q are isometries of ψ and φ, respectively, then $s_k^{\psi\varphi}(PAQ) = s_k^{\psi\varphi}(A)$.*

(4) *If ψ and φ are absolute, then the ψ,φ-s-numbers of a matrix do not change when we augment it with rows and columns of zeros.*

(5) *The interlacing or separation theorem: Let B $(m-r)\times(n-t)$ be a submatrix of A $m \times n$. If ψ is absolute and φ is arbitrary, then $s_k^{\psi\varphi}(A) \geq s_k^{\psi\varphi}(B)$. If φ is absolute, ψ is arbitrary, and s satisfies a certain technical condition (all the examples do), then $s_k^{\psi\varphi}(B) \geq s_{k+r+t}^{\psi\varphi}(A)$.*

Relations between Some s-Numbers

As pointed out in [4], we have the following relations, for all ψ, φ, A:

(1) $a_k^{\psi\varphi}(A) \geq g_k^{\psi\varphi}(A) \geq b_k^{\psi\varphi}(A)$, $a_k^{\psi\varphi}(A) \geq h_k^{\psi\varphi}(A) \geq m_k^{\psi\varphi}(A)$.

(2) $g_k^{\psi\varphi}(A) \geq m_k^{\psi\varphi}(A)$, $h_k^{\psi\varphi}(A) \geq b_k^{\psi\varphi}(A)$.

(3) $b_k^{\psi\varphi}(A) = [g_{n-k-1}^{\psi\varphi}(A^{-1})]^{-1}$ *if A $n \times n$ is invertible.*

(4) $a_k^{\chi_\infty\varphi}(A) = g_k^{\chi_\infty\varphi}(A)$ *(and, by duality, $a_k^{\psi\chi_1}(A) = h_k^{\psi\chi_1}(A)$).*

Computing s-Numbers

Evaluating s-numbers is usually a very difficult task. There are few general results. For the Euclidean norm, not unexpectedly, we have:

For any s, the $s_k^{\chi\chi}(A)$ *are the ordinary singular values of* A [4].

For arbitrary ψ and φ, almost nothing is known. A nice result (for the first s-number) is:

$S_{\psi\chi_1}(A) =$ *maximum of the* ψ*-norms of the columns of* A [2, 3].

We turn to *diagonal* matrices. For the case $\varphi = \psi$ we have:

If ψ *is absolute and* $A = diag(\alpha_1, \dots, \alpha_n)$, *with* $|\alpha_{\sigma(1)}| \geqslant \cdots \geqslant |\alpha_{\sigma(n)}|$, *then* $s_k^{\psi}(A) = |\alpha_{\sigma(k)}|$ *for any s.*

(If ψ is not absolute this is false.)

For $\varphi \neq \psi$ the situation is wild. We turn to the Hölder norms χ_p, $1 \leqslant p \leqslant \infty$, and we use the lighter notation s^{pq} instead of $s^{\chi_p \chi_q}$. The matrix A is still diagonal, $A = \mathrm{diag}(\alpha_1, \dots, \alpha_n)$, now with $\alpha_1 \geqslant \cdots \geqslant \alpha_n > 0$.

If $p < q$, *then* $a_k^{pq}(A) = g_k^{pq}(A) = h_k^{pq}(A) = (\sum_{j=k}^n \alpha_j^r)^{1/r}$, *where* $1/r = 1/p - 1/q$ [4].

For $p > q$ less is known. One small trick, using the previous result:

If $p > q$, *then* $b_k^{pq}(A) = [g_{n-k+1}^{qp}(A^{-1})]^{-1} = (\sum_{j=1}^k \alpha_j^r)^{1/r}$, *where* $1/r = 1/p - 1/q$ *(now* < 0*).*

Little else is known (see [5] for a detailed account and references). Let us be even more particular. Take $A = I_n$, $p = \infty$, $q = 1$, $\mathbf{F} = \mathbf{R}$, and consider the problem of evaluating $a_k^{\infty 1}(I_n)$, $k = 1, \dots, n$.

By previous remarks, $a_k^{\infty 1} = g_k^{\infty 1} = h_k^{\infty 1}$. Also, by the result in [2, 3] already mentioned, $S_{\infty 1}(M) =$ maximum of the ∞-norms of the columns of $M = \max_{i,j} |m_{ij}|$ for any M, whence trivially $a_1^{\infty 1}(I_n) = 1$.

Since here we have $1/p - 1/q = -1$, the Bernstein numbers are trivial: $b_k^{\infty 1}(I_n) = 1/k$. Therefore, $a_k^{\infty 1}(I_n) \geqslant 1/k$, $k = 2, \dots, n$. For $k = n$ there is equality, since $a_n^{\infty 1}(I_n) = \inf_{x \neq 0} \chi_\infty(x)/\chi_1(x) = 1/n$ (take all coordinates of x equal to 1).

The case $k = 2$ is also easy, because, taking all elements of X equal to $\frac{1}{2}$, we get $S_{\infty 1}(I_n - X) = \frac{1}{2}$, whence $a_2^{\infty 1}(I_n) \leqslant \frac{1}{2}$, and there must be equality.

To summarize, we have $a_1^{\infty 1}(I_n) = 1$, $a_2^{\infty 1}(I_n) = \frac{1}{2}$, $a_n^{\infty 1}(I_n) = 1/n$, and $a_k^{\infty 1}(I_n) \geqslant 1/k$, $k = 3, \dots, n-1$.

Recently, E. Marques de Sá [8] has found the value of $a_3^{\infty 1}(I_n)$. The numbers $a_4^{\infty 1}(I_n), \dots, a_{n-1}^{\infty 1}(I_n)$ remain unknown.

Sá's result is

$$a_3^{\infty 1}(I_n) = \frac{1}{1 + \sec(\pi/n)}.$$

This was already conjectured in [5]. The proof is a very long geometrical argument, beginning with the consideration of the *symmetric* case: evaluate the $S_{\infty 1}$-distance from I_n to the rank-2 symmetric matrices. This distance is precisely the number written above, and its determination reduces to finding the greatest minimum angle between n vectors pointing to the upper half plane in $\mathbf{R}^2$. The rest of the proof shows that the nonsymmetric matrices do not decrease the distance.

s-Numbers of Direct Sums

The problem to be addressed here is: what are the relations between the s-numbers of $C = A \dotplus B$ and those of A and B? For ordinary singular values, of course, the matter is trivial.

From now on we take $\varphi = \psi$, ψ absolute (otherwise, as suggested by what we said before, the problems become intractable).

We note the following: If $a_1 \geqslant a_2 \geqslant \cdots$ and $b_1 \geqslant b_2 \geqslant \cdots$ are two sequences of real numbers and if $c_1 \geqslant c_2 \geqslant \cdots$ is the sequence obtained from them by joining their elements, then $c_{i+j-1} \leqslant \max\{a_i, b_j\}$ and $c_{i+j} \geqslant \min\{a_i, b_j\}$. With this in mind, the following results have some interest. Assume ψ has the property that $\psi(u,v) \leqslant \psi(u',v')$ whenever $\psi(u) \leqslant \psi(u')$ and $\psi(v) \leqslant \psi(v')$, where u and u' (v and v') belong to the domain of A (B). Then:

(1) $s^{\psi}_{i+j-1}(A \dotplus B) \leqslant \max\{s^{\psi}_i(A), s^{\psi}_j(B)\}$ *for the approximation, Gelfand, and Kolmogorov numbers.*

(2) $s^{\psi}_{i+j}(A \dotplus B) \geqslant \min\{s^{\psi}_i(A), s^{\psi}_j(B)\}$ *for the Bernstein and Mitiagin numbers.*

Our interest in s-numbers of direct sums comes from the following. The converse of the interlacing theorem is in general false (see [7]). But something can be done in a particular situation: given $\alpha_1 \geqslant \beta_1 \geqslant \alpha_2 \geqslant \beta_2 \geqslant \cdots \geqslant \alpha_n \geqslant \beta_n \geqslant 0$, we want to prove that there exist $x_1, \ldots, x_n$ such that the matrix

$$\begin{bmatrix} \beta_1 & 0 & \cdots & 0 & x_1 \\ 0 & \beta_2 & \cdots & 0 & x_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \beta_n & x_n \end{bmatrix}$$

(or the transpose of this) has s-numbers $\alpha_1, \ldots, \alpha_n$. The argument is by induction on n. The case $n = 1$ is trivial. In the induction, we take in succession, in the matrix above, $x_1 = 0$, $x_2 = 0, \ldots, x_n = 0$. To use the induction hypothesis, which will ensure that all the boundary of the set $[\beta_1, +\infty) \times [\beta_2, \beta_1] \times \cdots \times [\beta_n, \beta_{n-1}]$ will be attained by s-numbers of matrices of the form above, we need the s-numbers of a matrix of the form $[\beta] \dotplus B$ to be β together with those of B. (For the interior, topological arguments are used, mainly the Brouwer degree.)

So we are interested in this particular problem: Let $A = [\beta] \dotplus B$. When are the s-numbers of A precisely β together with those of B?

The situation concerning this problem is rather strange. Let us look, for example, at the Gelfand numbers (there are similar results for other s-number functions).

Suppose $g_{k-1}^{\psi}(B) \geq \beta \geq g_k^{\psi}(B)$. *Then*:

(i) *For* $j = 1, \ldots, k-1$, $g_j^{\psi}(A) = g_j^{\psi}(B)$.
(ii) $g_k^{\psi}(A) \leq \beta$.
(iii) *For* $j = k+1, \ldots, n$, $g_j^{\psi}(A) \leq g_{j-1}^{\psi}(B)$.

In view of (ii) *and* (iii), *we have*:

(ii′) *If* $g_{k-1}^{\psi}(B) = b_{k-1}^{\psi}(B)$ *then* $g_k^{\psi}(A) = \beta$.
(iii′) *For* $j = k+1, \ldots, n$, *if* $g_{j-1}^{\psi}(B) = b_{j-1}^{\psi}(B)$ *then* $g_j^{\psi}(A) = g_{j-1}^{\psi}(B)$.

These strange conditions involving the Bernstein numbers lead us to still another question: For which norms ψ does the equality $g_k^{\psi}(A) = b_k^{\psi}(A)$ hold for all A and k?

Recall that $g_k^{\psi}(\cdot) \geq b_k^{\psi}(\cdot)$ always. This inequality can be strict. I am indebted to C. R. Johnson and Peter Nylen for the following example: Take $\psi = \chi_{\infty}$ and

$$A = \begin{bmatrix} 63 & 20 & 8 \\ 61 & 2 & 69 \\ 75 & 94 & 73 \end{bmatrix}.$$

Then $g_2^{\chi_\infty}(A) \approx 51$, $b_2^{\chi_\infty}(A) \approx 46.3$.

Differentiability of the norm seems to play a role, but it cannot be a sufficient condition for the equality, since by continuity the above matrix also serves as a counterexample for χ_p, p near ∞ (recall the first property we listed for s-number functions).

An intriguing possibility would be that the Gelfand and the Bernstein numbers coincide for all matrices (or operators) only in the Euclidean case. If this were so, the Courant-Fischer theorem would characterize the Euclidean norm among all norms. For related results, see [1], especially §§12–14.

Some of the topics treated here were studied at length in [6]. I thank the organizers for inviting me to give a talk at this meeting.

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HANKEL MATRICES FOR DISCRETE-TIME LINEAR PERIODIC SYSTEMS

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I. *Introduction*

Consider the discrete-time linear N-periodic system defined by the state-space model

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k), \\ y(k) &= C(k)x(k), \end{aligned} \tag{1}$$

where $A(k+N) = A(k) \in \mathbb{R}^{n \times n}$, $B(k+N) = B(k) \in \mathbb{R}^{n \times m}$, $C(k+N) = C(k) \in \mathbb{R}^{p \times n}$, $k \in \mathbb{Z}$, $N \in \mathbb{Z}^+$. The N-periodic system (1) is denoted by $(B(\cdot), A(\cdot), C(\cdot))_N$.

If we consider the initial state $x(s) = 0$, $s \in \mathbb{Z}$, the input-output application of the system (1), at time s, is given by

$$y(k+s) = \sum_{j=0}^{k-1} W_s(k, k-j)u(j+s), \qquad k \geqslant 1, \tag{2}$$

where

$$\begin{aligned} &W_s(k,j) = C(k+s)\Phi_A(k+s, k+s-j+1)B(k+s-j) \in \mathbb{R}^{p \times m}, \\ &s \in \mathbb{Z}, \quad k \geqslant 1, \quad j = 1, \dots, k, \end{aligned} \tag{3}$$

and $\Phi_A(\cdot,\cdot)$ is the transition matrix: $\Phi_A(k, k_0) = A(k-1)A(k-2)\cdots A(k_0)$ if $k > k_0$ and $\Phi_A(k_0, k_0) = I$.

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