A semidefinite approach to the K_i-cover problem

João Gouveia

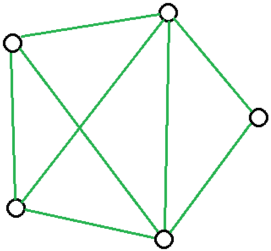
University of Coimbra

4th of August - SIAM AG 2013

with James Pfeiffer (U.Washington)

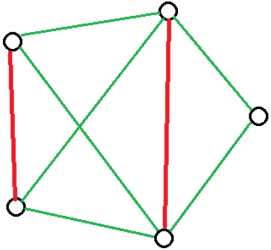
Triangle covers and Triangle-free sets

Given a graph $G = (V, E)$ we have:



Triangle covers and Triangle-free sets

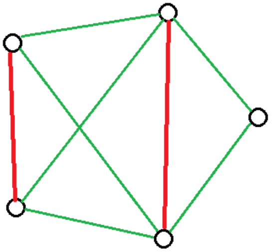
Given a graph $G = (V, E)$ we have:

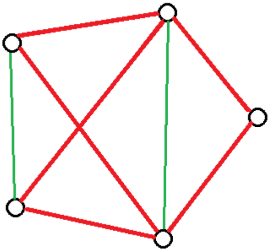


A triangle cover is a set of edges including at least one from each triangle

Triangle covers and Triangle-free sets

Given a graph $G = (V, E)$ we have:





A triangle cover is a set of edges including at least one from each triangle

A triangle-free subgraph is a set of edges not containing any triangle

These sets are complementary to each other.

Triangle-Cover Problem

Consider $G = (V, E)$ and edge-weights ω.

Triangle-Cover Problem

Consider $G = (V, E)$ and edge-weights ω.

Min Triangle Cover Problem

Find a triangle cover $T \subseteq E$ for which

$$\sum_{e \in T} \omega_e$$

is minimum.

Triangle-Cover Problem

Consider $G = (V, E)$ and edge-weights ω.

Min Triangle Cover Problem

Find a triangle cover $T \subseteq E$ for which

$$\sum_{e \in T} \omega_e$$

is minimum.

Max Triangle Free Problem

Find a triangle-free set $F \subseteq E$ for which

$$\sum_{e \in F} \omega_e$$

is maximum.

Triangle-Cover Problem

Consider $G = (V, E)$ and edge-weights ω.

Min Triangle Cover Problem

Find a triangle cover $T \subseteq E$ for which

$$\sum_{e \in T} \omega_e$$

is minimum.

Max Triangle Free Problem

Find a triangle-free set $F \subseteq E$ for which

$$\sum_{e \in F} \omega_e$$

is maximum.

Remarks:

- These problems are equivalent.

Triangle-Cover Problem

Consider $G = (V, E)$ and edge-weights ω.

Min Triangle Cover Problem

Find a triangle cover $T \subseteq E$ for which

$$\sum_{e \in T} \omega_e$$

is minimum.

Max Triangle Free Problem

Find a triangle-free set $F \subseteq E$ for which

$$\sum_{e \in F} \omega_e$$

is maximum.

Remarks:

- These problems are equivalent.
- These problems are NP-complete.

K_i-Cover Problem

Let G be any graph, and $\mathcal{K}_{i-1}$ its set of cliques of size $i-1$.

K_i-Cover Problem

Let G be any graph, and $\mathcal{K}_{i-1}$ its set of cliques of size $i-1$.

Min K_i-Cover Problem

Find a K_i-cover $T \subseteq \mathcal{K}_{i-1}$ for which

$$\sum_{k \in T} \omega_k$$

is minimum.

K_i-Cover Problem

Let G be any graph, and $\mathcal{K}_{i-1}$ its set of cliques of size $i-1$.

Min K_i-Cover Problem

Find a K_i-cover $T \subseteq \mathcal{K}_{i-1}$ for which

$$\sum_{k \in T} \omega_k$$

is minimum.

Max Triangle Free Problem

Find a K_i-free set $F \subseteq \mathcal{K}_{i-1}$ for which

$$\sum_{k \in F} \omega_k$$

is maximum.

K_i-Cover Problem

Let G be any graph, and $\mathcal{K}_{i-1}$ its set of cliques of size $i-1$.

Min K_i-Cover Problem

Find a K_i-cover $T \subseteq \mathcal{K}_{i-1}$ for which

$$\sum_{k \in T} \omega_k$$

is minimum.

Max Triangle Free Problem

Find a K_i-free set $F \subseteq \mathcal{K}_{i-1}$ for which

$$\sum_{k \in F} \omega_k$$

is maximum.

Remarks:

K_i-Cover Problem

Let G be any graph, and $\mathcal{K}_{i-1}$ its set of cliques of size $i-1$.

Min K_i-Cover Problem

Find a K_i-cover $T \subseteq \mathcal{K}_{i-1}$ for which

$$\sum_{k \in T} \omega_k$$

is minimum.

Max Triangle Free Problem

Find a K_i-free set $F \subseteq \mathcal{K}_{i-1}$ for which

$$\sum_{k \in F} \omega_k$$

is maximum.

Remarks:

- For $i = 3$ is the triangle cover problem.

K_i-Cover Problem

Let G be any graph, and $\mathcal{K}_{i-1}$ its set of cliques of size $i-1$.

Min K_i-Cover Problem

Find a K_i-cover $T \subseteq \mathcal{K}_{i-1}$ for which

$$\sum_{k \in T} \omega_k$$

is minimum.

Max Triangle Free Problem

Find a K_i-free set $F \subseteq \mathcal{K}_{i-1}$ for which

$$\sum_{k \in F} \omega_k$$

is maximum.

Remarks:

- For $i = 3$ is the triangle cover problem.
- For $i = 2$ is the stable set problem.

K_i-Cover Problem

Let G be any graph, and $\mathcal{K}_{i-1}$ its set of cliques of size $i-1$.

Min K_i-Cover Problem

Find a K_i-cover $T \subseteq \mathcal{K}_{i-1}$ for which

$$\sum_{k \in T} \omega_k$$

is minimum.

Max Triangle Free Problem

Find a K_i-free set $F \subseteq \mathcal{K}_{i-1}$ for which

$$\sum_{k \in F} \omega_k$$

is maximum.

Remarks:

- For $i = 3$ is the triangle cover problem.
- For $i = 2$ is the stable set problem.
- All are NP-complete [Comforti-Corneil-Mahjoub]

Triangle-Free Polytope

Given a graph $G = (\{1, ..., n\}, E)$ we define $P_3(G)$, the triangle-free polytope of G, in the following way:

Triangle-Free Polytope

Given a graph $G = (\{1, ..., n\}, E)$ we define $P_3(G)$, the triangle-free polytope of G, in the following way:

- For every triangle-free set $F \subseteq E$ consider its characteristic vector $\chi_F \in \{0, 1\}^E$;

Triangle-Free Polytope

Given a graph $G = (\{1, ..., n\}, E)$ we define $P_3(G)$, the triangle-free polytope of G, in the following way:

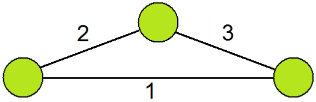
- For every triangle-free set $F \subseteq E$ consider its characteristic vector $\chi_F \in \{0, 1\}^E$;
- let $S_3 \subset \{0, 1\}^n$ be the collection of all those vectors;

Triangle-Free Polytope

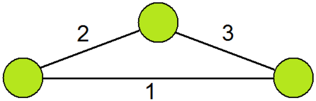
Given a graph $G = (\{1, ..., n\}, E)$ we define $P_3(G)$, the triangle-free polytope of G, in the following way:

- For every triangle-free set $F \subseteq E$ consider its characteristic vector $\chi_F \in \{0, 1\}^E$;
- let $S_3 \subset \{0, 1\}^n$ be the collection of all those vectors;
- the polytope $P_3(G)$ is then defined as the convex hull of the vectors in S_3.

Example

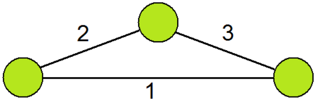


Example

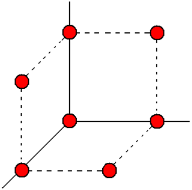


$$S_G = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,0),(0,1,1),(1,0,1)\}$$

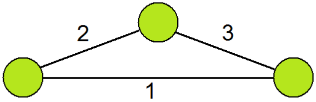
Example



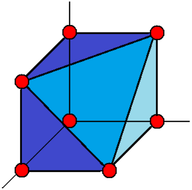
$$S_G = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,0),(0,1,1),(1,0,1)\}$$



Example



$$S_G = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,0),(0,1,1),(1,0,1)\}$$



Reformulation

Triangle-Free Problem Reformulated

Given a graph $G = (V, E)$ and a weight vector $\omega \in \mathbb{R}^E$, solve

$$\mu(G, \omega) := \max_{x \in P_3(G)} \langle \omega, x \rangle .$$

Reformulation

Triangle-Free Problem Reformulated

Given a graph $G = (V, E)$ and a weight vector $\omega \in \mathbb{R}^E$, solve

$$\mu(G, \omega) := \max_{x \in P_3(G)} \langle \omega, x \rangle .$$

However finding $P_3(G)$ is as hard as solving the original problem.

Reformulation

Triangle-Free Problem Reformulated

Given a graph $G = (V, E)$ and a weight vector $\omega \in \mathbb{R}^E$, solve

$$\mu(G, \omega) := \max_{x \in P_3(G)} \langle \omega, x \rangle .$$

However finding $P_3(G)$ is as hard as solving the original problem.

We intend to find approximations for it.

Fractional Triangle-Free Polytope

The simplest relaxation of $P_3(G)$ is the fractional triangle free polytope of G, $\mathrm{FRAC}_\Delta(G)$, the set defined by the following inequalities.

Fractional Triangle-Free Polytope

The simplest relaxation of $P_3(G)$ is the fractional triangle free polytope of G, $\mathrm{FRAC}_\Delta(G)$, the set defined by the following inequalities.

- $1 \geq x_i \geq 0$ for $i \in E$ ($0 - 1$ constrains);

Fractional Triangle-Free Polytope

The simplest relaxation of $P_3(G)$ is the fractional triangle free polytope of G, $\mathrm{FRAC}_\Delta(G)$, the set defined by the following inequalities.

- $1 \geq x_i \geq 0$ for $i \in E$ $(0 - 1$ constrains);
- $x_i + x_j + x_k \leq 2$ for all $\{i, j, k\} \in \mathcal{K}_3$ (triangle constrains).

Fractional Triangle-Free Polytope

The simplest relaxation of $P_3(G)$ is the fractional triangle free polytope of G, $\mathrm{FRAC}_\Delta(G)$, the set defined by the following inequalities.

- $1 \geq x_i \geq 0$ for $i \in E$ $(0-1$ constrains);
- $x_i + x_j + x_k \leq 2$ for all $\{i,j,k\} \in \mathcal{K}_3$ (triangle constrains).

It is possible to optimize over this polytope in polynomial time.

Fractional Triangle-Free Polytope

The simplest relaxation of $P_3(G)$ is the fractional triangle free polytope of G, $\mathrm{FRAC}_{\Delta}(G)$, the set defined by the following inequalities.

- $1 \geq x_i \geq 0$ for $i \in E$ $(0 - 1$ constrains);
- $x_i + x_j + x_k \leq 2$ for all $\{i, j, k\} \in \mathcal{K}_3$ (triangle constrains).

It is possible to optimize over this polytope in polynomial time.

Krivelevich [1995] proved for the unweighted triangle cover version an approximation factor of 2.

Fractional Triangle-Free Polytope

The simplest relaxation of $P_3(G)$ is the fractional triangle free polytope of G, $\mathrm{FRAC}_\Delta(G)$, the set defined by the following inequalities.

- $1 \geq x_i \geq 0$ for $i \in E$ ($0 - 1$ constrains);
- $x_i + x_j + x_k \leq 2$ for all $\{i, j, k\} \in \mathcal{K}_3$ (triangle constrains).

It is possible to optimize over this polytope in polynomial time.

Krivelevich [1995] proved for the unweighted triangle cover version an approximation factor of 2.

We want to use moment matrices to approximate this problem.

Triangle Ideal and sums of squares approximations

The polynomials vanishing on S_3 are those in the ideal

$$I_3 = \left\langle x_e x_f x_g,\ x_i^2 - x_i : \forall \text{ triangles } \{e, f, g\},\ \forall i \in E \right\rangle.$$

Triangle Ideal and sums of squares approximations

The polynomials vanishing on S_3 are those in the ideal

$$I_3 = \left\langle x_e x_f x_g,\ x_i^2 - x_i : \forall \text{ triangles } \{e, f, g\},\ \forall i \in E \right\rangle .$$

$f \in \mathbb{R}[x]$ is k-sos modulo I_3 if and only if

$$f \equiv ({h_1}^2 + {h_2}^2 + \ldots + {h_m}^2) \mod I_3,$$

for some polynomials $h_1, \ldots, h_m$ with degree less or equal k.

Triangle Ideal and sums of squares approximations

The polynomials vanishing on S_3 are those in the ideal

$$I_3 = \left\langle x_e x_f x_g,\ x_i^2 - x_i : \forall \text{ triangles } \{e, f, g\},\ \forall i \in E \right\rangle .$$

$f \in \mathbb{R}[x]$ is k-sos modulo I_3 if and only if

$$f \equiv ({h_1}^2 + {h_2}^2 + ... + {h_m}^2) \quad \text{mod } I_3,$$

for some polynomials $h_1, ..., h_m$ with degree less or equal k.

Theta Bodies of an ideal

$$\mathrm{TH}_d(I_3) = \bigcap_{\ell \text{ linear }, \ell \ k\text{-sos modulo } I_3} \{\mathbf{x} \in \mathbb{R}^E : \ell(\mathbf{x}) \geq 0\}$$

Triangle Ideal and sums of squares approximations

The polynomials vanishing on S_3 are those in the ideal

$$I_3 = \left\langle x_e x_f x_g,\ x_i^2 - x_i : \forall \text{ triangles } \{e, f, g\},\ \forall i \in E \right\rangle .$$

$f \in \mathbb{R}[x]$ is k-sos modulo I_3 if and only if

$$f \equiv ({h_1}^2 + {h_2}^2 + ... + {h_m}^2) \quad \text{mod } I_3,$$

for some polynomials $h_1, ..., h_m$ with degree less or equal k.

Theta Bodies of an ideal

$$\mathrm{TH}_d(I_3) = \bigcap_{\ell \text{ linear }, \ell\ k\text{-sos modulo } I_3} \{\mathbf{x} \in \mathbb{R}^E : \ell(\mathbf{x}) \geq 0\}$$

We have $P_3(G) \subseteq \cdots \subseteq \mathrm{TH}_3(I_3) \subseteq \mathrm{TH}_2(I_3) \subseteq \mathrm{FRAC}_\Delta(G)$.

Facets of $P_3(G)$

Comforti-Corneil-Mahjoub catalogued some families of facets for the polytope of the K_3-free problem. Among them:

1. Binary inequalities: $0 \leq x_k \leq 1$, for all $k \in E$,

Facets of $P_3(G)$

Comforti-Corneil-Mahjoub catalogued some families of facets for the polytope of the K_3-free problem. Among them:

1. Binary inequalities: $0 \leq x_k \leq 1$, for all $k \in E$,

2. Triangle inequalities: $x_i + x_j + x_k \leq 2$ for all $\{i, j, k\} \in \mathcal{K}_3$

Facets of $P_3(G)$

Comforti-Corneil-Mahjoub catalogued some families of facets for the polytope of the K_3-free problem. Among them:

1. Binary inequalities: $0 \leq x_k \leq 1$, for all $k \in E$,

2. Triangle inequalities: $x_i + x_j + x_k \leq 2$ for all $\{i, j, k\} \in \mathcal{K}_3$

3. Δ-p-hole inequalities:

Facets of $P_3(G)$

Comforti-Corneil-Mahjoub catalogued some families of facets for the polytope of the K_3-free problem. Among them:

1. Binary inequalities: $0 \leq x_k \leq 1$, for all $k \in E$,

2. Triangle inequalities: $x_i + x_j + x_k \leq 2$ for all $\{i, j, k\} \in \mathcal{K}_3$

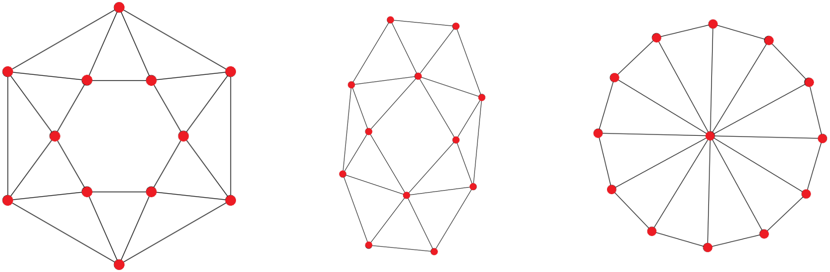
3. Δ-p-hole inequalities: **????**

Δ-p holes

A Δ-p-hole is a graph made up of p copies of K_3, $C_1, C_2, \cdots, C_p$ such that C_k and C_j share an edge if and only if $|k-j| \equiv 1$.

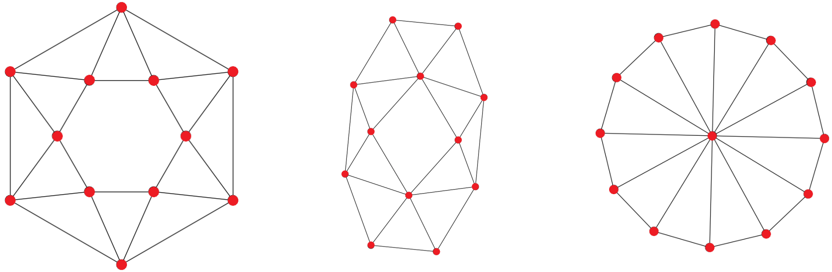
Δ-p holes

A Δ-p-hole is a graph made up of p copies of K_3, $C_1, C_2, \cdots, C_p$ such that C_k and C_j share an edge if and only if $|k - j| \equiv 1$.



Δ-p holes

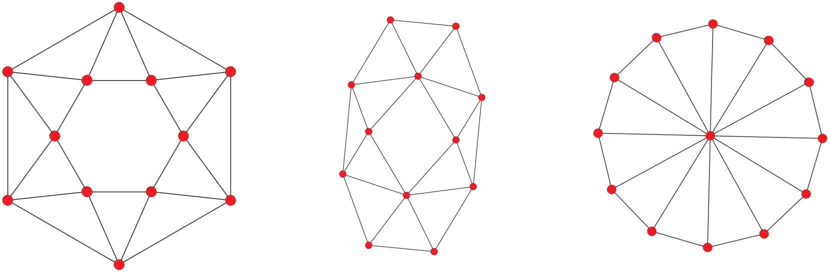
A Δ-p-hole is a graph made up of p copies of K_3, $C_1, C_2, \cdots, C_p$ such that C_k and C_j share an edge if and only if $|k-j| \equiv 1$.



In particular wheels of order Δ-p-holes.

Δ-p holes

A Δ-p-hole is a graph made up of p copies of K_3, $C_1, C_2, \cdots, C_p$ such that C_k and C_j share an edge if and only if $|k-j| \equiv 1$.



In particular wheels of order Δ-p-holes.

If p odd, and $H \subseteq G$ a Δ-p-hole, $P_3(G)$ has a facet:

$$\sum_H x_j \leq 3(\frac{p-1}{2}) + 1.$$

Separation

Comforti-Corneil-Mahjoub give polytime separation algorithm for several families of facets, including **binary**, **clique** and **wheel** inequalities, thus providing a polytime algorithm to optimize over them.

They couldn't give a separation algorithm for the more general **Δ-p-hole** inequalities.

Separation

Comforti-Corneil-Mahjoub give polytime separation algorithm for several families of facets, including **binary**, **clique** and **wheel** inequalities, thus providing a polytime algorithm to optimize over them.

They couldn't give a separation algorithm for the more general **Δ-p-hole** inequalities.

Let $Q(G)$ be the polytope cut out by **binary**, **clique** and **Δ-p-hole** inequalities.

Separation

Comforti-Corneil-Mahjoub give polytime separation algorithm for several families of facets, including **binary**, **clique** and **wheel** inequalities, thus providing a polytime algorithm to optimize over them.

They couldn't give a separation algorithm for the more general **Δ-p-hole** inequalities.

Let $Q(G)$ be the polytope cut out by **binary**, **clique** and **Δ-p-hole** inequalities.

Containment

$$P_3(G) \subseteq \mathrm{TH}_2(I_3) \subseteq Q(G).$$

Separation

Comforti-Corneil-Mahjoub give polytime separation algorithm for several families of facets, including **binary**, **clique** and **wheel** inequalities, thus providing a polytime algorithm to optimize over them.

They couldn't give a separation algorithm for the more general **Δ-p-hole** inequalities.

Let $Q(G)$ be the polytope cut out by **binary**, **clique** and **Δ-p-hole** inequalities.

Containment

$$P_3(G) \subseteq \mathrm{TH}_2(I_3) \subseteq Q(G).$$

General Containment

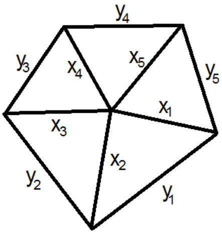
$$P_i(G) \subseteq \mathrm{TH}_{\lceil i/2 \rceil}(I_i) \subseteq Q(G).$$

Proof by example

Enough to give an sos certificate.

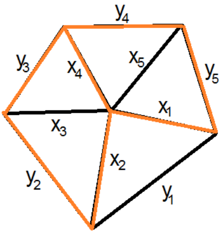
Proof by example

Enough to give an sos certificate.



Proof by example

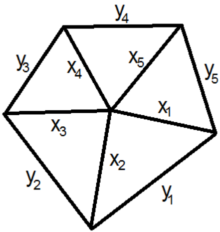
Enough to give an sos certificate.



$$7 - \sum x_i - \sum y_i =$$

Proof by example

Enough to give an sos certificate.



$$7-\sum x_i-\sum y_i = \begin{array}{l}(1-y_1-x_1x_2)^2+(1-y_2-x_2x_3)^2\\ +(1-y_3-x_3x_4)^2+(1-y_4-x_5x_5)^2\\ +(1-y_5-x_5x_1)^2\\ +(1-x_1-x_2-x_3+x_1x_2+x_2x_3+x_1x_3)^2\\ +(1-x_3-x_4-x_5+x_3x_4+x_3x_5+x_4x_5)^2\\ +(x_3-x_3x_1-x_3x_5+x_1x_5)^2\end{array}$$

Properties of the relaxation (triangle-case)

Using the relation between triangle free graphs and cuts, and a result by Laurent we get

Convergence limitations

$P_3(K_n) \subsetneq \mathrm{TH}_i(I_3)$ for all $i < (n-2)/4$.

Properties of the relaxation (triangle-case)

Using the relation between triangle free graphs and cuts, and a result by Laurent we get

Convergence limitations

$P_3(K_n) \subsetneq \mathrm{TH}_i(I_3)$ for all $i < (n-2)/4$.

Let τ be the triangle cover number of G. We can approximate it by

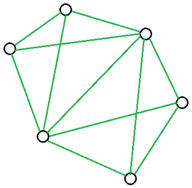
$$\tau^{\dagger} = |E| - max_{x \in \mathrm{TH}_2(I_3)} \langle x, \mathbb{1} \rangle .$$

Approximation ratio

For all G we have $2\tau^{\dagger}(G) \geq \tau(G) \geq \tau^{\dagger}(G)$.

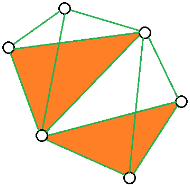
Tuza's Conjecture

Let $G = (V, E)$ be a graph and $\nu(G)$ be its triangle packing number.



Tuza's Conjecture

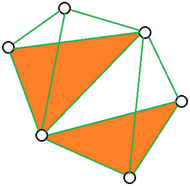
Let $G = (V, E)$ be a graph and $\nu(G)$ be its triangle packing number.



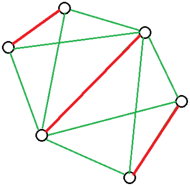
$\nu(G) = 2$

Tuza's Conjecture

Let $G = (V, E)$ be a graph and $\nu(G)$ be its triangle packing number.



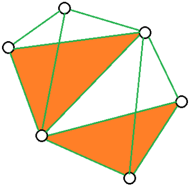
$\nu(G) = 2$



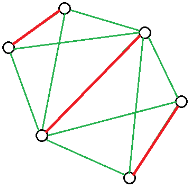
$\tau(G) = 3$

Tuza's Conjecture

Let $G = (V, E)$ be a graph and $\nu(G)$ be its triangle packing number.



$\nu(G) = 2$



$\tau(G) = 3$

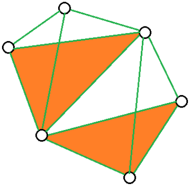
Tuza's Conjecture

$\tau(G) \leq 2\nu(G)$

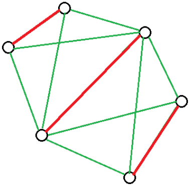
Note that $\tau(G) \leq 3\nu(G)$ is trivial.

Tuza's Conjecture

Let $G = (V, E)$ be a graph and $\nu(G)$ be its triangle packing number.



$\nu(G) = 2$



$\tau(G) = 3$

Tuza's Conjecture

$\tau(G) \leq 2\nu(G)$

Note that $\tau(G) \leq 3\nu(G)$ is trivial.

$\tau^{\text{frac}}(G) \leq 2\nu(G)$ [Krivelevich], is it true that $\tau^{\dagger}(G) \leq 2\nu(G)$?

The end

Thank You