Semidefinite Representations

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CMUC - Universidade de Coimbra

19th March - Seminário - CELC - Universidade de Lisboa

1. Spectrahedra and SDP

Semidefinite Programming

An SDP problem is an optimization problem of the form

$$\max_{x} c^t x \text{ s.t. } A_0 + A_1 x_1 + \ldots + A_n x_n \succeq 0.$$

Here, A_i's are symmetric real matrices.

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These convex problems can be solved efficiently, and their geometry very rich. Particularly, a lot of interest has been focused on their feasible sets.

Representability

Definition

We say a set $S \subseteq \mathbb{R}^n$ is a **spectrahedron** (or **LMI-representable**) if there exist symmetric matrices $A_0, ..., A_n$ such that

$$S = \{\mathbf{x} \in \mathbb{R}^n : A_0 + A_1 x_1 + ... + A_n x_n \succeq 0\}.$$

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LMI and SDP representable sets are necessarily convex and semialgebraic, but what other conditions do they have to satisfy?

PSD cone and spectrahedra

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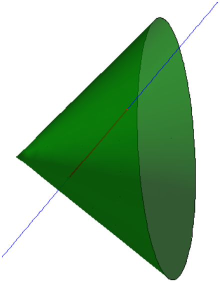

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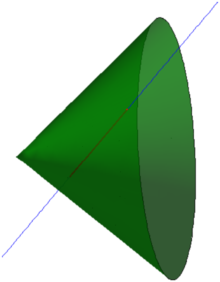
Optimizing over projections of spectrahedra can be done efficiently.



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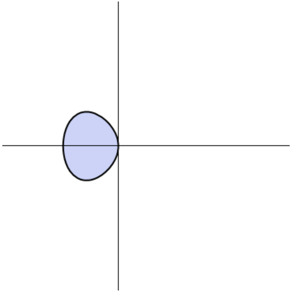

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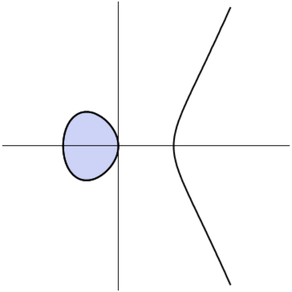

$x^3 - x - y^2$

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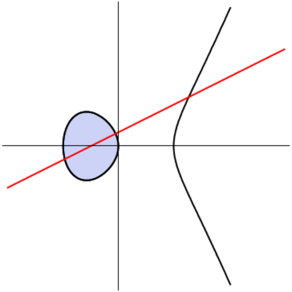

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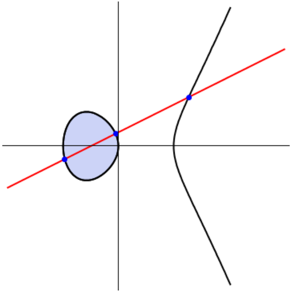

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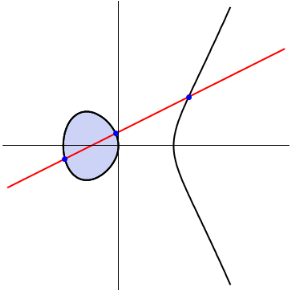

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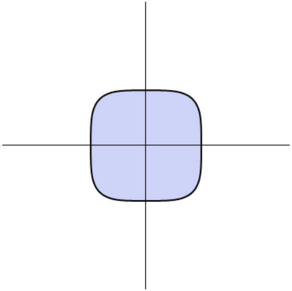

$x^4 + y^4 - 1$ - Not Real Zero

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The big question: Is every convex semialgebraic set SDP-representable?

2. Theta Bodies

Convex Hulls of Algebraic Sets

Problem

Given an algebraic set

$$\{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) = \ldots = g_m(\mathbf{x}) = 0\},$$

we want to find a good "convex" description for its convex hull.

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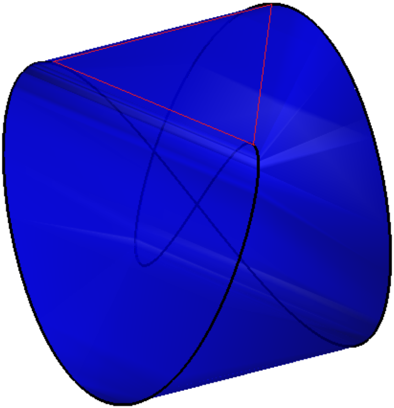
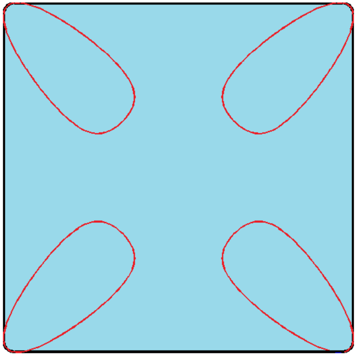
Notation:

- $I = \langle g_1, \ldots, g_m \rangle$,
- $\mathcal{V}_{\mathbb{R}}(I) = \{\text{Real zeros of } I\}$.

Examples



$I = \left\langle x^2 - y^2 - xz, z - 4x^3 + 3x \right\rangle$



$I = \left\langle 25(x^4 + y^4 + 1) - 34(x^2y^2 + x^2 + y^2) \right\rangle$

Theta body

Convex Hull

$$\mathrm{cl}(\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))) = \bigcap_{\ell \text{ linear }, \ell|_{\mathcal{V}_{\mathbb{R}}(I)} \geq 0} \{x \in \mathbb{R}^n : \ell(x) \geq 0\}$$

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We can replace $\ell|_{\mathcal{V}_{\mathbb{R}}(I)} \geq 0$ by ℓ being sos modulo I:

$$\ell \equiv \sum_i h_i^2 + I.$$

If $deg(h_i) \leq k$ we say that ℓ is k-sos.

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Definition

$$\mathrm{TH}_k(I) := \bigcap_{\ell \text{ linear }, \ell \text{ } k\text{-sos modulo } I} \{x \in \mathbb{R}^n : \ell(x) \geq 0\}$$

Theta body - Example

(Loading...)

$$\mathrm{TH}_2(I) \text{ for } I = \langle x(x^2+y^2) - x^4 - x^2y^2 - y^4 \rangle.$$

Convergence

$$\mathrm{TH}_1(I) \supseteq \mathrm{TH}_2(I) \supseteq \ldots \supseteq \mathrm{TH}_k(I) \supseteq \mathrm{cl}(\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I)))$$

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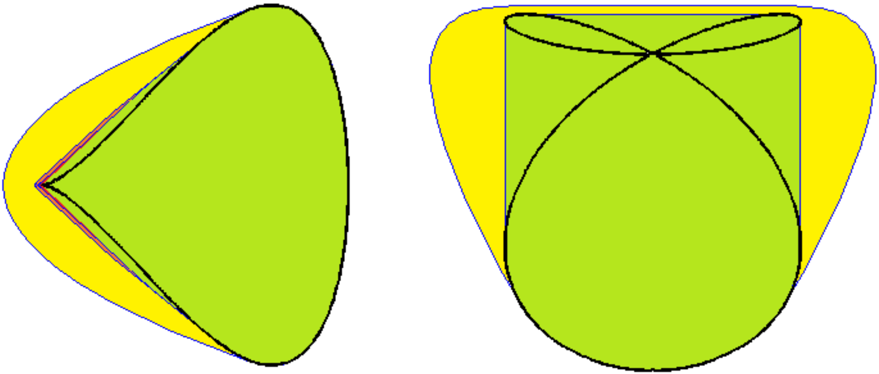
G-Netzer

If $\mathcal{V}_{\mathbb{R}}(I)$ has "bad" singularities, that convergence is not finite.

Examples



Two quartics and their theta body sequence.

Finite sets

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G-Thomas

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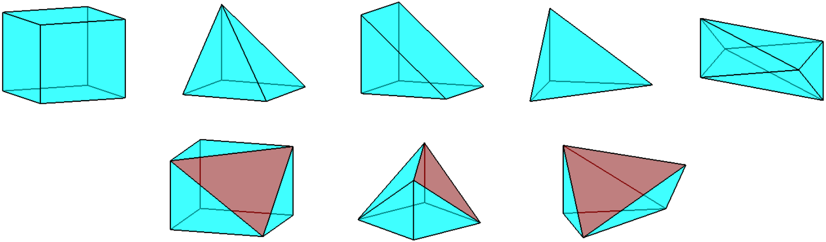
G-Parrilo-Thomas

If $S \subseteq \mathbb{R}^n$ is finite, $I(S)$ is TH_1-exact if and only if S is the set of vertices of a 2-level polytope.

2-level polytopes



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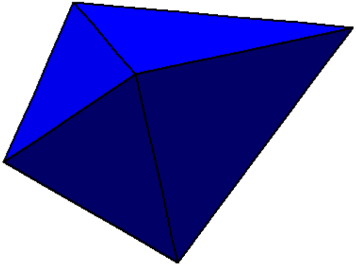

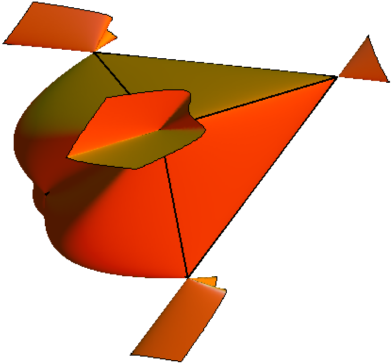

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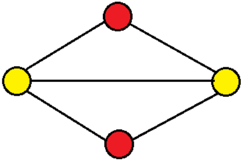
The stable set problem.



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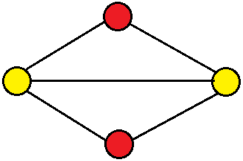
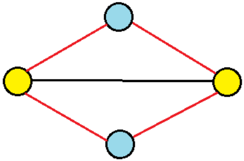


G-Laurent-Parrilo-Thomas

The max-cut problem.



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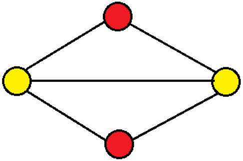

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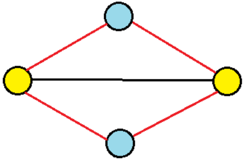
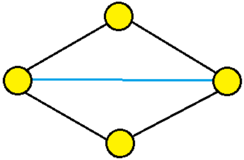


G-Thomas

The max triangle-free subgraph / min K_3-cover problem.



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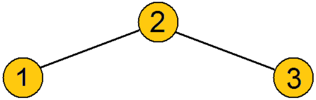
STAB(G) - stable set polytope of G.

Example



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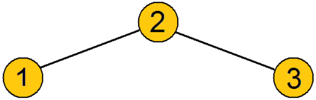

$$S_G = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,0,1)\}$$

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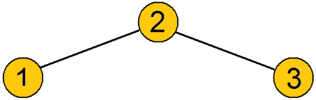

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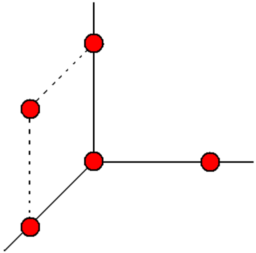

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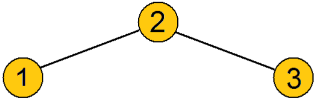

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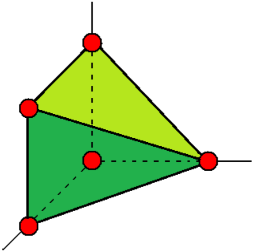

Theta body for stable set

Given a graph G with n nodes, TH_1 is the set of all vectors $x \in \mathbb{R}^n$ such that

$$\begin{bmatrix} 1 & x^t \\ x & U \end{bmatrix} \succeq 0$$

for some symmetric $U \in \mathbb{R}^{n \times n}$ with $\mathrm{diag}(U) = x$ and $U_{ij} = 0$ for all edges (i, j).

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Theorem (Lovász)

$\mathrm{TH}_1 = \mathrm{STAB}(G)$ *if and only if G is perfect.*

Lifts of stable sets

If G is perfect STAB(G) is a projection of a slice of the cone PSD_{n+1}.
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We want to frame all these approaches and their limits in one single theory

3. Lifts of Convex Sets

Lifts of Polytopes

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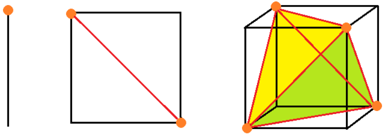

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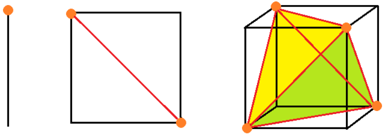

The polytope P_n has 2^{n-1} vertices (one per odd set) and 2^{n-1} facets (one per even set).

Parity Polytope

There is a much shorter description.

PP_n is the set of $\mathbf{x} \in \mathbb{R}^n$ such that there exists for every odd $1 \leq k \leq n$ a vector $\mathbf{z}_k \in \mathbb{R}^n$ and a real number α_k such that

- $\sum_k \mathbf{z}_k = \mathbf{x}$;
- $\sum_k \alpha_k = 1$;
- $\| \mathbf{z}_k \|_1 = k\, \alpha_k$;
- $0 \leq (\mathbf{z}_k)_i \leq \alpha_k$.

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$O(n^2)$ variables and $O(n^2)$ constraints.

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Canonical LP Lift

Given a polytope P, a canonical LP lift is a description

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for some affine space L and affine map Φ. We say it is a $\mathbb{R}_+^k$-lift.

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We are interested in the smallest k such that P has a $\mathbb{R}^k_+$-lift, a much better measure of "LP-complexity" .

Two definitions

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Slack Matrix

The slack matrix of P is the matrix $S_P \in \mathbb{R}^{v \times f}$ defined by

$$S_P(i,j) = h_j(p_i).$$

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Let P be a polytope with facets defined by $h_1(\mathbf{x}) \geq 0, \ldots, h_f(\mathbf{x}) \geq 0$, and vertices $p_1, \ldots, p_v$.

Slack Matrix

The slack matrix of P is the matrix $S_P \in \mathbb{R}^{v \times f}$ defined by

$$S_P(i,j) = h_j(p_i).$$

Nonnegative Factorization

Given a nonnegative matrix $M \in \mathbb{R}_+^{n \times m}$ we say that it has a k-nonnegative factorization, or a $\mathbb{R}_+^k$-factorization if there exist matrices $A \in \mathbb{R}_+^{n \times k}$ and $B \in \mathbb{R}_+^{k \times m}$ such that

$$M = A \cdot B.$$

Yannakakis' Theorem

Theorem (Yannakakis 1991)

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- Does LP solve all polynomial combinatorial problems?

The Hexagon

Consider the regular hexagon.

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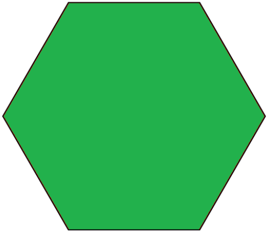

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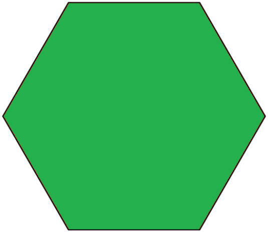

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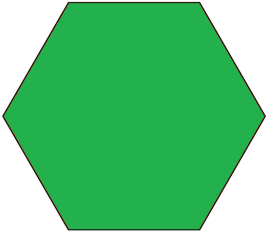

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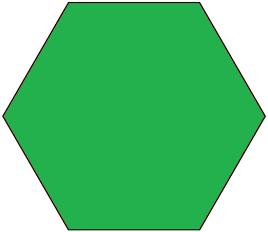

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Hexagon - continued

It is the projection of the slice of $\mathbb{R}^5_+$ cut out by

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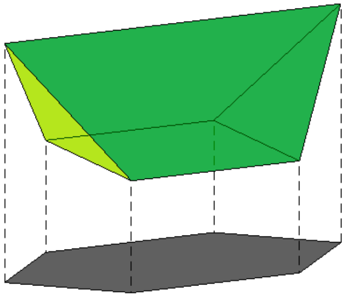

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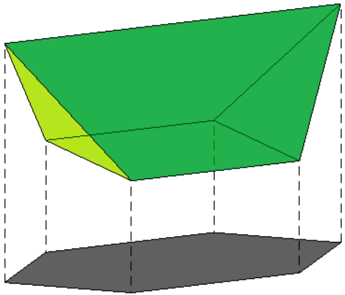

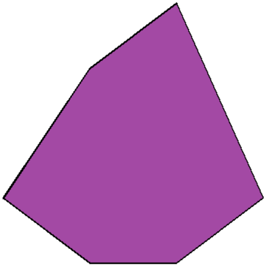

For irregular hexagons a $\mathbb{R}^6_+$-lift is the only we can have.

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Note that if the theta body is exact, it is a PSD-lift.

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Given a nonnegative matrix $M \in \mathbb{R}_+^{n \times m}$ we say that it has a K-factorization if there exist $a_1, \ldots a_n \in K$ and $b_1, \ldots, b_m \in K^*$ such that

$$M_{i,j} = \langle a_i, b_j \rangle .$$

We can now generalize Yannakakis.

Generalized Yannakakis for polytopes

Theorem (G-Parrilo-Thomas)

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- That is actually the best possible PSD-lift.
- **[Burer]** In general STAB(G) has a CP_{n+1}-lift.
- We can generalize Yannakakis further to other convex sets by introducing a slack operator.

Open questions - further directions

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- Are there polynomial sized LP-lifts for the stable set polytope of a perfect graph?
- Which sets are SDP-representable, i.e., which sets have SDP-lifts?

The end

Thank You