From approximate factorizations to approximate lifts

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Linear Lifts

A linear lift of a polytope P of size k is a description

$$P = \left\{ x \in \mathbb{R}^n \;\middle|\; \exists y \text{ s.t. } a_0 + \sum a_i x_i + \sum b_i y_i \geq 0 \right\}$$

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Equivalently, it is a polytope Q with k facets such that $L(Q) = P$ for some affine map L.

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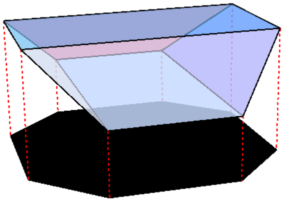

Linear lift of size 6

Slack Matrix

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1	−1	−2	−2	−1	1	2	2
2	2	1	−1	−2	−2	−1	1

$$2 - x \geq 0$$
$$3 - x - y \geq 0$$
$$2 - y \geq 0$$
$$3 + x - y \geq 0$$
$$2 + x \geq 0$$
$$3 + x + y \geq 0$$
$$2 + y \geq 0$$
$$3 - x + y \geq 0$$

$$\begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

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$2 - x \geq 0$	1	3	4	4	3	1	0	0
$3 - x - y \geq 0$								
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$2 - x \geq 0$	1	3	4	4	3	1	0	0
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$2 - x \geq 0$	1	3	4	4	3	1	0	0
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$3 + x - y \geq 0$	2	0	0	2	4	6	6	4
$2 + x \geq 0$	3	1	0	0	1	3	4	4
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$2 + y \geq 0$	4	4	3	1	0	0	1	3
$3 - x + y \geq 0$	4	6	6	4	2	0	0	2

Nonnegative Factorizations

Nonnegative Factorization

Given a nonnegative matrix $M \in \mathbb{R}_+^{n \times m}$ a k-nonnegative factorization, is a pair of matrices $A \in \mathbb{R}_+^{k \times n}$ and $B \in \mathbb{R}_+^{k \times m}$ such that

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The slack matrix of a regular octagon has nonnegative rank 6

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More precisely, let $P = \{x : H^t x \leq \mathbb{1}\}$ and $S_P = A^t \cdot B$ be a k-nonnegative factorization.

$$P = \left\{ x \in \mathbb{R}^n \ : \ \exists y \in \mathbb{R}^k_+ \text{ s.t. } H^t x + A^t y = \mathbb{1} \right\}$$

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This formulation is very overdetermined, any perturbation of A makes it unfeasible. We need a more robust version.

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Let $P = \{x : H^t x \leq \mathbb{1}\}$ and V be the matrix whose columns are the vertices of P.

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These are robust formulations, but too big.

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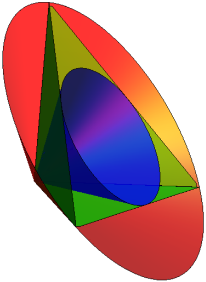

Then $\mathcal{O}^n_{in} \subseteq \mathbb{R}^n_+ \subseteq \mathcal{O}^n_{out}$, and furthermore, $(\mathcal{O}^n_{in})^* = (\mathcal{O}^n_{out})$.

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If $S_P = A^t \cdot B$ is a k-nonnegative factorization then

$$P = \mathrm{Inn}_P(A) = \left\{x \in \mathbb{R}^n \,:\, \exists y \in \mathbb{R}^k_+ \text{ s.t. } \mathbb{1} - H^t x - A^t y \in \mathcal{O}^f_{in}\right\}$$

$$P = \mathrm{Out}_P(B) = \left\{Vz \,:\, z \in \mathcal{O}^v_{out}, \quad \mathbb{1}^t z \leq 1, \quad Bz \in \mathbb{R}^k_+\right\}$$

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$$P = \mathrm{Out}_P(B) = \left\{ Vz \; : \; z \in \mathcal{O}^v_{out}, \quad \mathbb{1}^t z \leq 1, \quad Bz \in \mathbb{R}^k_+ \right\}$$

Both $\mathrm{Inn}_P(A)$ and $\mathrm{Out}_P(B)$ are actually $\mathbb{R}^k_+ \times SOC_{k+n+1}$ lifts, so we gain robustness and don't loose effectiveness.

Containment

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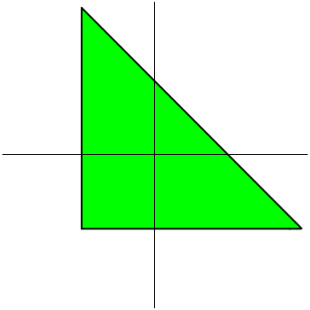
Polar Property

$(\mathrm{Inn}_P(A))^\circ = \mathrm{Out}_{P^\circ}(A)$ and $(\mathrm{Out}_P(B))^\circ = \mathrm{Inn}_{P^\circ}(B)$.

Example

$$\text{Let } P = \left\{ (x, y) : \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}^t \begin{bmatrix} x \\ y \end{bmatrix} \leq \mathbb{1} \right\}$$



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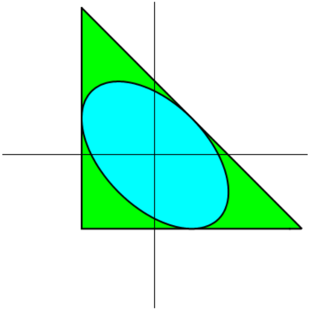
$$\text{Inn}_P(0) = \left\{ (x,y) \; : \; 3(x+y)^2 + (x-y)^2 \le 3 \right\}$$



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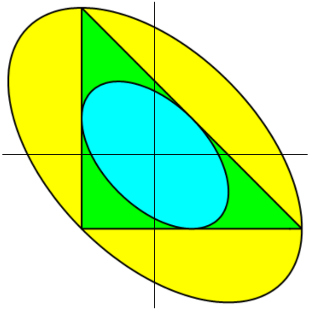

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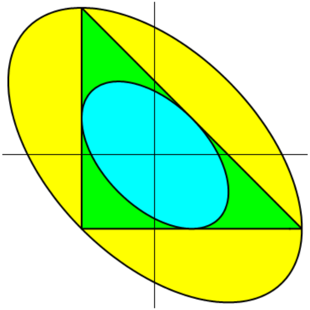
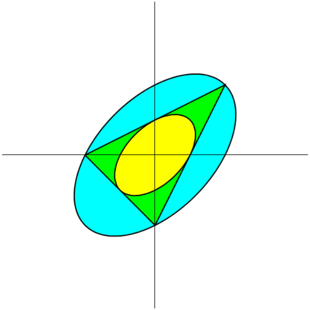
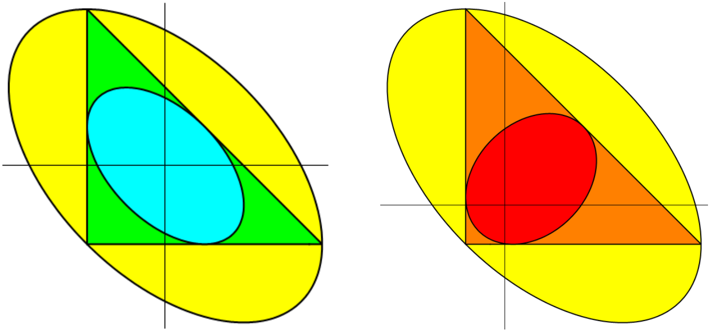




Translation (In)variance

Note that the inner approximations depend on the choice of the center, while the outer is invariant.



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Then

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Let $\tilde{S} = A^t \cdot B$, and P a polytope such that

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Then

$$\frac{1}{1+\varepsilon_1} P \subseteq \mathrm{Inn}_P(B) \subseteq P; \quad P \subseteq \mathrm{Out}_P(A) \subseteq (1+\varepsilon_2)P.$$

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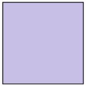
Good factorizations give good approximations.

Example

Consider P the square with vertices $(\pm 1, \pm 1)$.



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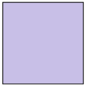

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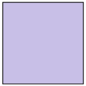

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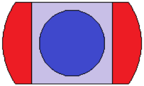

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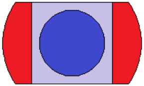

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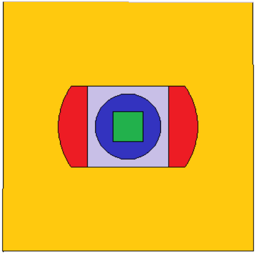

Further thoughts

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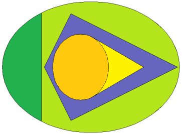

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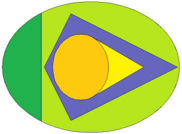

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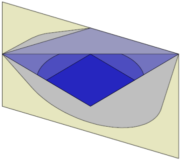

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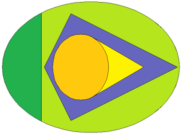

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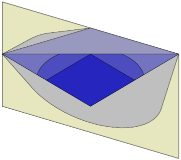

- Approximate lifts to approximate factorizations is easy.

THE END

THANK YOU