Semidefinite lifts of polytopes

João Gouveia

CMUC - Universidade de Coimbra

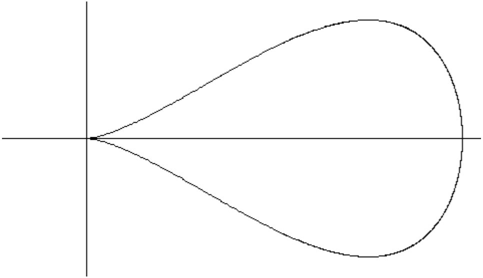
20th April 2013 - CMUC workshop

Background - Polynomial Optimization

Linear Optimization over Algebraic Varieties

Maximize linear $l(x)$ over all x such that

$$x \in S = \{x \mid p_i(x) = 0,\ i = 1, ..., t\}.$$



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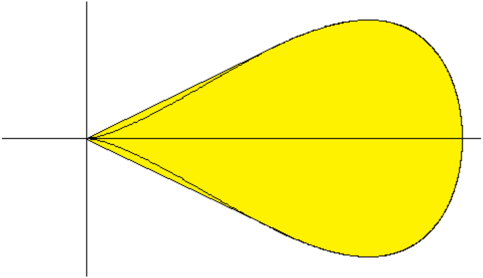

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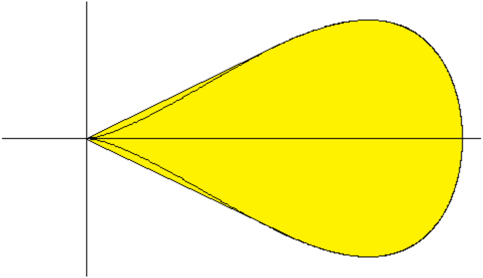

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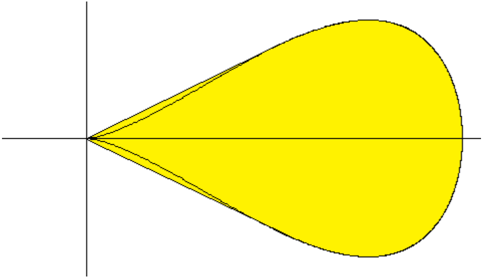

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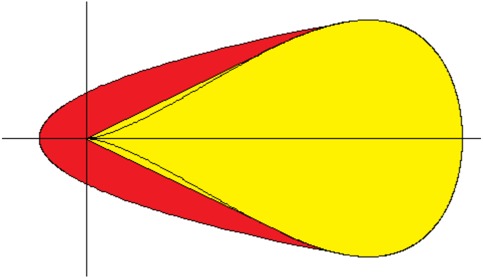

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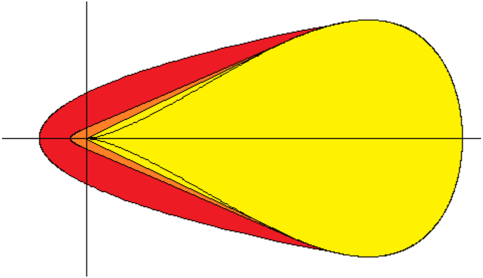

$$x_1{}^4 - x_1{}^3 + x_2{}^2 = 0, \ k = 3$$

Combinatorial Optimization

Many combinatorial problems can be stated in this way.

Max stable set

Given a graph $G = (V, E)$, find the maximum stable set.

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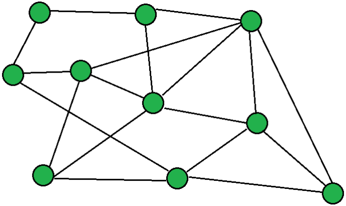

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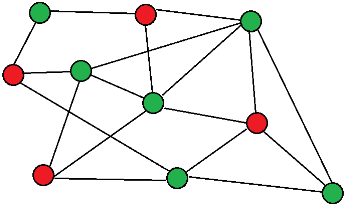

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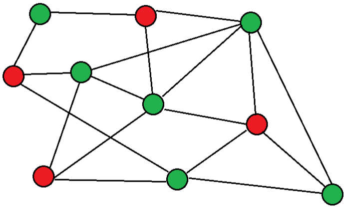

Max stable set - reformulation

Maximize $\sum_{i \in V} x_i$ over all x verifying

$$x_i^2 - x_i = 0, \forall\, i \in V; \quad x_i x_j = 0, \forall \{i, j\} \in E.$$

Why do we do this?

Main Motivation

$$TH_k(\mathcal{I}) = \{x \mid h(x) \geq 0, \ \forall h \text{ linear s.t. } h \equiv_{\mathcal{I}} \sum q_i^2, \ \deg(q_i) \leq k\}$$

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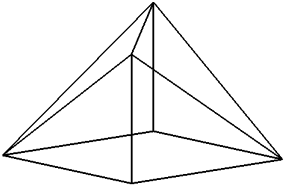

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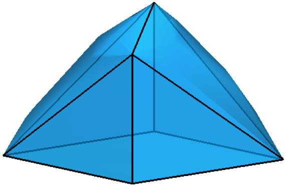
$$\mathrm{TH}_1 = \left\{ (x, y, z) \mid \exists w_1, w_2, \begin{bmatrix} 1 & x & y & z \\ x & x & w_1 & w_2 \\ y & w_1 & y & w_2 \\ z & w_2 & w_2 & z \end{bmatrix} \succeq 0 \right\}.$$



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In combinatorial optimization, classes of graphs where we have **exactness at a fixed degree** correspond to classes of graphs where we can solve the problem in **poly-time**.

However, this is only one possible way to obtain good representations. **What can we say in general?**

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PSD factorizations

A PSD factorization of M of size k is a collection of $k \times k$ psd matrices $A_1, \cdots, A_m$ and $B_1, \cdots, B_m$ such that $M_{i,j} = \langle A_i, B_j \rangle$.

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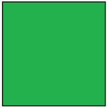

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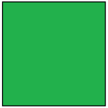

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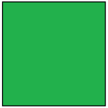

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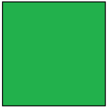

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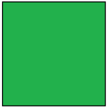

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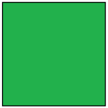

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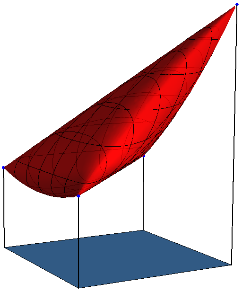

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