Approximating Convex Hulls of Planar Quartics

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Semialgebraic sets

Given a set of polynomials $\mathcal{G} = \{g_1, \ldots, g_m\}$ we denote

$$S(\mathcal{G}) = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}.$$

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We are particularly interested in the case

$$S(g) = \{x \in \mathbb{R}^n : g(x) \geq 0\}.$$

for some simple instances of g.

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A certificate of nonnegativity on $S(\mathcal{G})$ for p is given by a representation

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The set of all such polynomials where $\deg(\sigma_0)$ and $\deg(\sigma_i g_i)$ are less or equal to $2k$ is denoted by $\mathrm{QM}_k(\mathcal{G})$.

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We can therefore relax it by

$$\mathcal{L}_k(\mathcal{G}) = \bigcap_{\ell \text{ linear }, \ell \in \mathrm{QM}_k(\mathcal{G})} \{x \in \mathbb{R}^n : \ell(x) \geq 0\}$$

which we call the *k-th Lasserre Relaxation* of $S(\mathcal{G})$.

Lasserre Relaxation - Example

(Loading...)

$\mathcal{L}_2(g)$ for $g = x(x^2 + y^2) - x^4 - x^2y^2 - y^4$.

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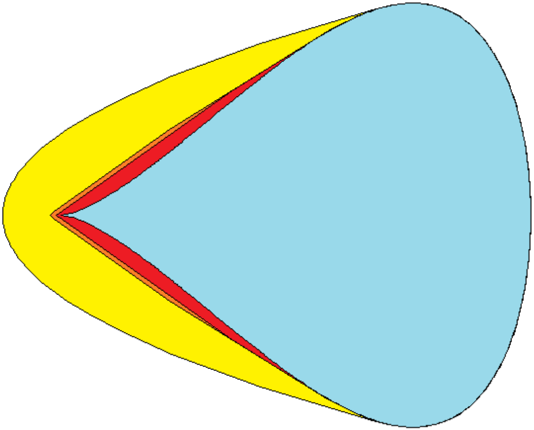
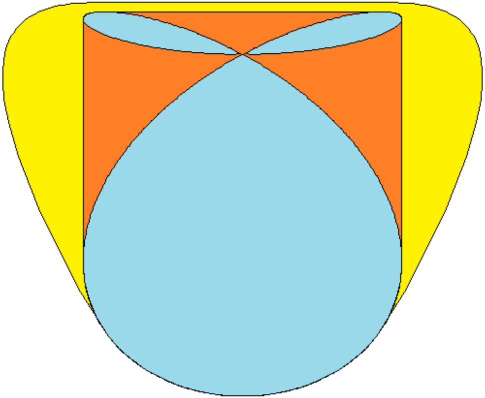
The paper also contains a list of beautiful examples. This talk is in the same spirit.

Example 1



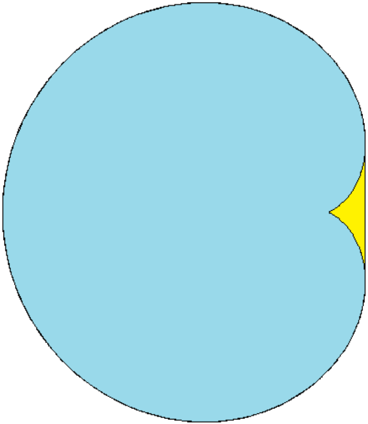
Relaxations $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ of $S(-x^4 + x^3 - y^2)$.

Example 2



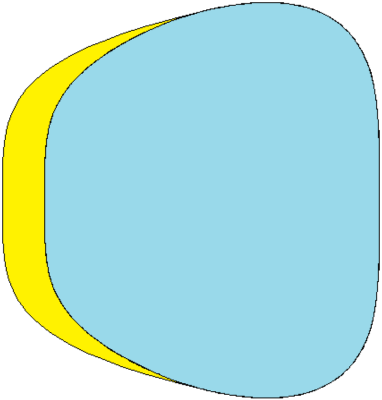
Relaxations $\mathcal{L}_2$ and $\mathcal{L}_3$ for $S(-y(y^2 - x^2) - (x^2 + y^2)^2)$.

Example 3



Relaxation $\mathcal{L}_2$ of $S(4(x^2 + y^2) - (x^2 + y^2 + 2x)^2)$.

Example 4



Relaxation $\mathcal{L}_2$ of $S(x + x^2 - 2x^4 - y^4)$.

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Smooth quartic revisited

(Loading...)

$\mathcal{L}_2(g_\alpha)$ for $g_\alpha = (1-\alpha)x + x^2 - 2x^4 - y^4$, with $\alpha \in [0,1]$.

Polynomial Shadow

Given a polynomial $p(x_1, \ldots, x_n)$, consider its graph

$$G_p = \{(x_0, x) \in \mathbb{R}^{n+1} \ : \ x_0 = p(x_1, \ldots, x_n)\}.$$

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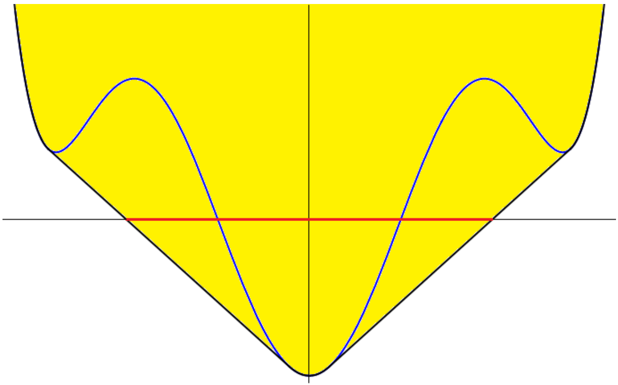
The shadow of p, $\mathrm{Sh}(p) \subseteq \mathbb{R}^n$, is the set

$$\mathrm{Sh}(p) = \{x \in \mathbb{R}^n \;:\; (0, x) \in \mathrm{conv}(G_p)\}.$$

Example



Graph of p and Sh(p) for $p = -4 + 7x^2 - 2x^4 + 1/6x^6$.

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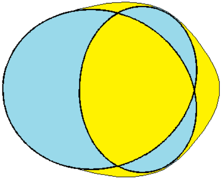
Hence $\ell(x) \geq 0$ is valid over $\mathrm{Sh}(p)$. □

Scarabeus

Consider $p = (x^2 - y^2)^2 - (x^2 + y^2)(x^2 + y^2 + 4x)^2$. This cuts out the scarabeus sixtic which does not seem $\mathcal{L}_3$-exact.



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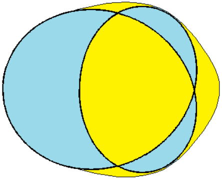

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$$p(-4,0) = -256, \;\; p(1,0) = 24 \;\; \Rightarrow \;\; \left(\frac{4}{7}, 0, 0\right) \in \mathrm{conv}(G_p).$$

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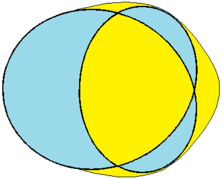

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However, $\max_{x \in S(p)} = \left(-50 + 11\sqrt{22}\right)/27 \approx 0.06$.

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Actually the proof gives us a better result.

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If p has n variables and degree $2d$ where

- $n = 1$;
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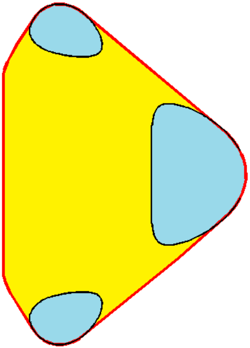
In particular, if p is a concave planar quartic, $\mathcal{L}_2(p) = S(p)$.

Example



$$\mathcal{L}_2(20x^4 + 85y^2x^2 - 25x^2 - 3x + 20y^4 - 25y^2 + 7).$$

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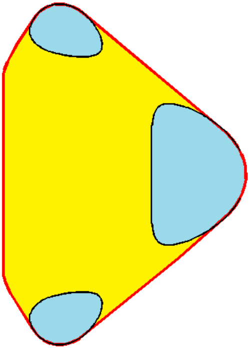

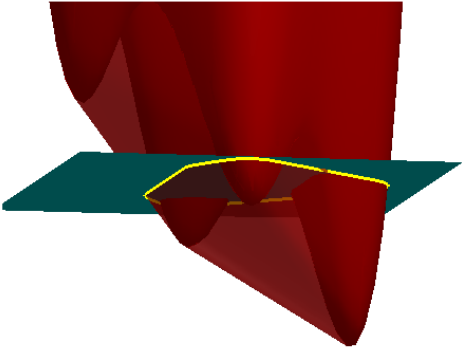

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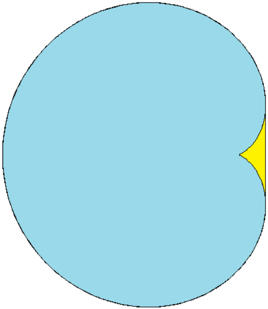

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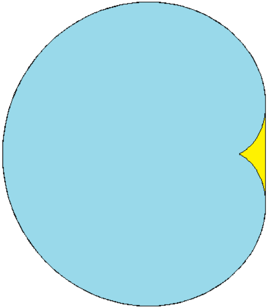

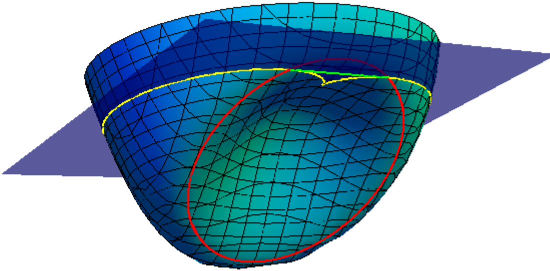

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Univariate quartics

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$\mathcal{L}_2(p) = \text{conv}(S(p))$ iff $\{b_1, b_2\} \subseteq \text{conv}(\{a_1, a_2, a_3, a_4\})$.

Otherwise $\mathcal{L}_2(p) = \text{conv}(\{a_1, a_2, a_3, a_4, c\}$, where c is the x-intercept of the bitangent at $\{b_1, b_2\}$.

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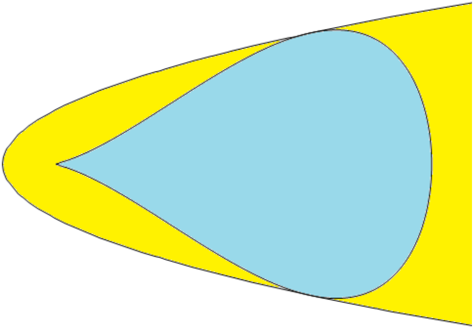
Eliminating all variables but w we should get a curve containing the boundary of $\mathcal{L}_2(p)$.
Unfortunately it has not worked well in practice.

Teardrop revisited



For the teardrop curve $x^3 - x^4 - y^2$ we get the boundary curve

$$-1 - 8x + 64y^2.$$

The end

Thank You