From approximate factorizations to approximate lifts

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Linear Lifts

A linear lift of a polytope P of size k is a description

$$P = \left\{ x \in \mathbb{R}^n \;\middle|\; \exists y \text{ s.t. } a_0 + \sum a_i x_i + \sum b_i y_i \geq 0 \right\}$$

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Equivalently, it is a polytope Q with k facets such that $L(Q) = P$ for some affine map L.

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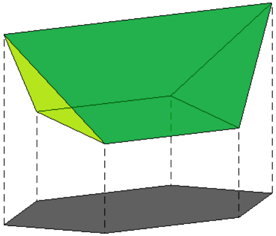

Linear lift of size 5

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Let P be a polytope with facets given by $h_1(x) \geq 0, \ldots, h_f(x) \geq 0$, and vertices $p_1, \ldots, p_v$.

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Nonnegative Factorizations

Nonnegative Factorization

Given a nonnegative matrix $M \in \mathbb{R}_+^{n \times m}$ a k-nonnegative factorization, is a pair of matrices $A \in \mathbb{R}_+^{k \times n}$ and $B \in \mathbb{R}_+^{k \times m}$ such that

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The slack matrix of an hexagon has nonnegative rank 5

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More precisely, let $P = \{x : H^t x \leq \mathbb{1}\}$ and $S_P = A^t \cdot B$ be a k-nonnegative factorization.

$$P = \left\{ x \in \mathbb{R}^n \ : \ \exists y \in \mathbb{R}^k_+ \text{ s.t. } H^t x + A^t y = \mathbb{1} \right\}$$

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This formulation is very overdetermined, any perturbation of A makes it unfeasible. We need a more robust version.

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These are robust formulations, but too big.

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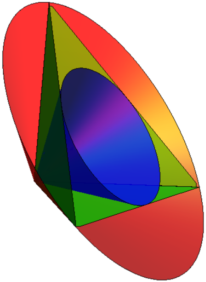

Then $\mathcal{O}_{in}^{n} \subseteq \mathbb{R}_{+}^{n} \subseteq \mathcal{O}_{out}^{n}$, and furthermore, $(\mathcal{O}_{in}^{n})^{*} = (\mathcal{O}_{out}^{n})$.

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Both $\mathrm{Inn}_P(A)$ and $\mathrm{Out}_P(B)$ are actually $\mathbb{R}^k_+ \times SOC_{k+n+1}$ lifts, so we gain robustness and don't loose effectiveness.

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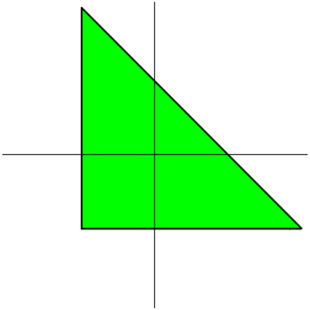
Polar Property

$(\mathrm{Inn}_P(A))^\circ = \mathrm{Out}_{P^\circ}(A)$ and $(\mathrm{Out}_P(B))^\circ = \mathrm{Inn}_{P^\circ}(B)$.

Example

Let $P = \left\{ (x, y) : \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}^t \begin{bmatrix} x \\ y \end{bmatrix} \leq \mathbb{1} \right\}$



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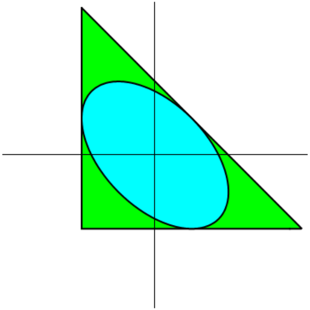
$$\text{Inn}_P(0) = \left\{ (x, y) \ : \ 3(x + y)^2 + (x - y)^2 \leq 3 \right\}$$



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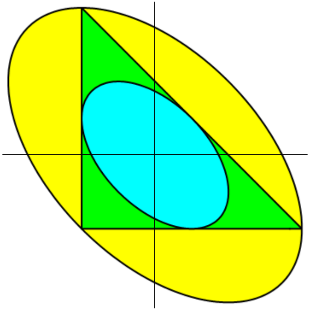

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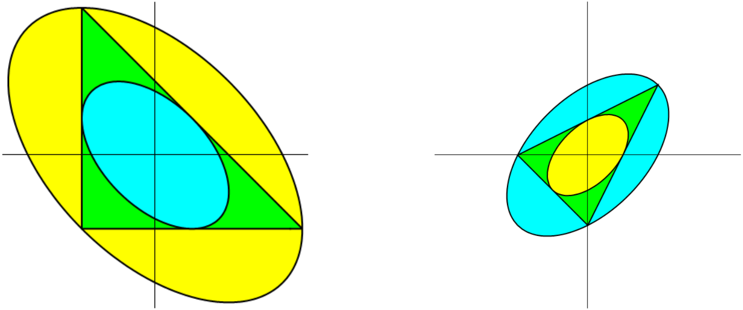
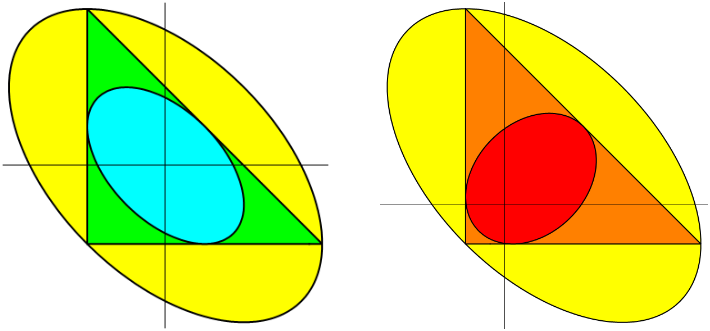


Translation (In)variance

Note that the inner approximations depend on the choice of the center, while the outer is invariant.



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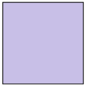
Good factorizations give good approximations.

Example

Consider P the square with vertices $(\pm 1, \pm 1)$.



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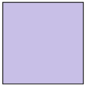
$$S_P = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$



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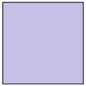

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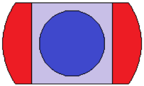

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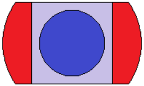
$$\varepsilon_1 = 2/3\sqrt{10}; \qquad \varepsilon_2 = 2/3\sqrt{6}$$



Example

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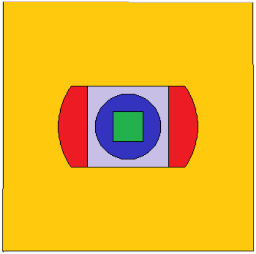
$$S_P = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \\ 2 & 0 & 0 & 2 \end{bmatrix} \qquad \tilde{S} = A^t \cdot B = \begin{bmatrix} 4/3 & 4/3 & 4/3 & 0 \\ 4/3 & 4/3 & 4/3 & 0 \\ 4/3 & 0 & 4/3 & 4/3 \\ 4/3 & 0 & 4/3 & 4/3 \end{bmatrix}$$

$$\varepsilon_1 = 2/3\sqrt{10}; \qquad \varepsilon_2 = 2/3\sqrt{6}$$



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- **More on the session tomorrow afternoon: WED.C.11**

THE END

THANK YOU