Sums of squares relaxations for Convex Hulls of Semialgebraic Sets

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Acknowledgments

Work done with:

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Pablo Parrilo - MIT

Tim Netzer - University of Leipzig

Optimization over algebraic sets

Problem

$$\begin{aligned} \max_{x \in \mathbb{R}^n} \quad & \langle c, x \rangle \\ \text{s.t.} \quad & h_i(x) = 0, \quad i = 1, \cdots, k, \end{aligned}$$

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Let $I = \langle h_1, ..., h_k \rangle$ and $\mathcal{V}_{\mathbb{R}}(I)$ the set of real solutions of the system. We can rewrite the problem as

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$$\begin{aligned} \min_{\lambda \in \mathbb{R}} \quad & \lambda \\ \text{s.t.} \quad & \lambda - \langle c, x \rangle \geq 0, \quad \forall x \in \mathcal{V}_{\mathbb{R}}(I). \end{aligned}$$

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Let $f \in \mathbb{R}[x]$ we say that f is **sos modulo** I if there are polynomials $p_i \in \mathbb{R}[x]$ such that

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- Verifying the k-sos property can be done efficiently given a 'nice' basis for the quotient space $\mathbb{R}[x]/I$.
- We can now approximate the original program by a hierarchy of relaxations.

Sums of squares relaxations

Recall we had the problem

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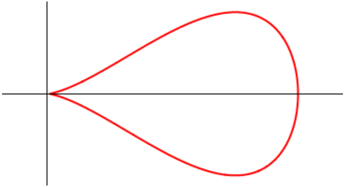
We can now replace it by the hierarchy

Problem-SOS$_k$

$$\begin{aligned} \min_{\lambda \in \mathbb{R}} \quad & \lambda \\ \text{s.t.} \quad & \lambda - \langle c, x \rangle \text{ is } k\text{-sos modulo } I. \end{aligned}$$

Example

Consider the ideal $I = \langle x^4 - x^3 + y^2 \rangle$ and its variety.



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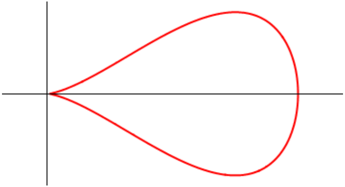

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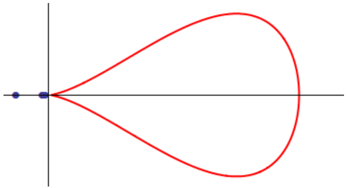

We want to minimize x in this variety. The relaxation gives us

	$k = 1$	$k = 2$	$k = 3$	$k = 4$
λ_{sos_k}	$-\infty$	$-.1250$	$-.0208$	$-.0091$

.

Theta Bodies

The geometric set underlying these relaxations is called the k-**th theta body** of the ideal I.

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$$\mathrm{TH}_k(I) := \bigcap_{\ell \text{ linear }, \ell \ k\text{-sos modulo } I} \{x \in \mathbb{R}^n : \ell(x) \geq 0\}$$

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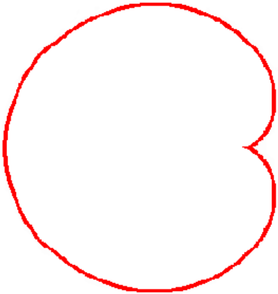
Convex Hull

$$\mathrm{cl}(\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))) = \bigcap_{\ell \text{ linear }, \ell|_{\mathcal{V}_{\mathbb{R}}(I)} \geq 0} \{x \in \mathbb{R}^n : \ell(x) \geq 0\}$$

Example 2

Consider the cardioid given by
$I = \langle (x^2 + 2x + y^2)^2 - 4(x^2 + y^2) \rangle$.



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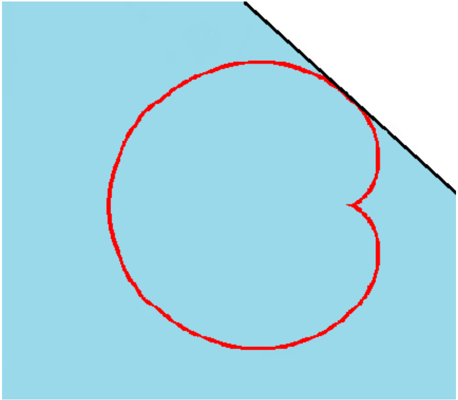

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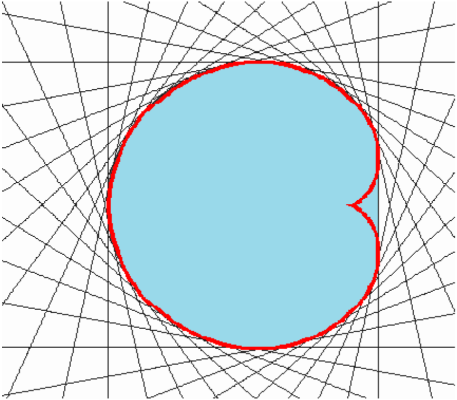

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be a basis of $\mathbb{R}[x]/I$ and $\mathcal{B}_k = \{f_i : \deg(f_i) \leq k\}$ for all k.

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$$(f^k(x))(f^k(x))^t = \sum_{f_i \in \mathcal{B}} A_i f_i(x)$$

for some symmetric matrices A_i.

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$$(f^k(x))(f^k(x))^t = \sum_{f_i \in \mathcal{B}} A_i f_i(x)$$

for some symmetric matrices A_i. Given a vector y indexed by the elements in $\mathcal{B}$ we define the ***k*-th truncated combinatorial moment matrix** of y as

$$M_{\mathcal{B},k}(y) = \sum_{f_i \in \mathcal{B}} A_i y_{f_i}.$$

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Let $I = \langle x_1^2 - x_1, x_2^2 - x_2, x_3^2 - x_3 \rangle \subset \mathbb{R}[x_1, x_2, x_3]$,

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$M_{\mathcal{B},1}(y)$ is given by:

	1	x_1	x_2	x_3	x_1x_2	x_1x_3	x_2x_3	$x_1x_2x_3$
1	$\mathbf{y_0}$	$\mathbf{y_1}$	$\mathbf{y_2}$	$\mathbf{y_3}$	y_{12}	y_{13}	y_{23}	y_{123}
x_1	$\mathbf{y_1}$	$\mathbf{y_1}$	$\mathbf{y_{12}}$	$\mathbf{y_{13}}$	y_{12}	y_{13}	y_{123}	y_{123}
x_2	$\mathbf{y_2}$	$\mathbf{y_{12}}$	$\mathbf{y_2}$	$\mathbf{y_{23}}$	y_{12}	y_{123}	y_{23}	y_{123}
x_3	$\mathbf{y_3}$	$\mathbf{y_{13}}$	$\mathbf{y_{23}}$	$\mathbf{y_3}$	y_{123}	y_{13}	y_{23}	y_{123}
x_1x_2	y_{12}	y_{12}	y_{12}	y_{123}	y_{12}	y_{123}	y_{123}	y_{123}
x_1x_3	y_{13}	y_{13}	y_{123}	y_{13}	y_{123}	y_{13}	y_{123}	y_{123}
x_2x_3	y_{23}	y_{123}	y_{23}	y_{23}	y_{123}	y_{123}	y_{23}	y_{123}
$x_1x_2x_3$	y_{123}	y_{123}	y_{123}	y_{123}	y_{123}	y_{123}	y_{123}	y_{123}

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1	y_0	y_1	y_2	y_3	y_{12}	y_{13}	y_{23}	y_{123}
x_1	y_1	y_1	y_{12}	y_{13}	y_{12}	y_{13}	y_{123}	y_{123}
x_2	y_2	y_{12}	y_2	y_{23}	y_{12}	y_{123}	y_{23}	y_{123}
x_3	y_3	y_{13}	y_{23}	y_3	y_{123}	y_{13}	y_{23}	y_{123}
x_1x_2	y_{12}	y_{12}	y_{12}	y_{123}	y_{12}	y_{123}	y_{123}	y_{123}
x_1x_3	y_{13}	y_{13}	y_{123}	y_{13}	y_{123}	y_{13}	y_{123}	y_{123}
x_2x_3	y_{23}	y_{123}	y_{23}	y_{23}	y_{123}	y_{123}	y_{23}	y_{123}
$x_1x_2x_3$	y_{123}	y_{123}	y_{123}	y_{123}	y_{123}	y_{123}	y_{123}	y_{123}

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Note that for all $p = (p_1, ..., p_n) \in \mathcal{V}_{\mathbb{R}}(I)$ we have

$$M_{\mathcal{B},k}(f^k(p)) = \sum_{f_i \in \mathcal{B}_k} A_i f_i(p) = (f^k(p))(f^k(p))^t \succeq 0.$$

Relation between approaches

We say that an ideal I is real radical if

$$I = \sqrt[\mathbb{R}]{I} := \{p : -p^{2m} \text{ sos modulo } I \text{ for some } m\}$$

or equivalently if $I = \mathcal{I}(\mathcal{V}_{\mathbb{R}}(I))$ (Real Nullstellensatz).

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Theorem (G.,Thomas)

Fix an ideal I and an integer k. Then there exists $m_k \in \mathbb{N}$ such that $\mathrm{TH}_{m_k}(I) \subseteq \mathrm{TH}_k(\sqrt[\mathbb{R}]{I})$.

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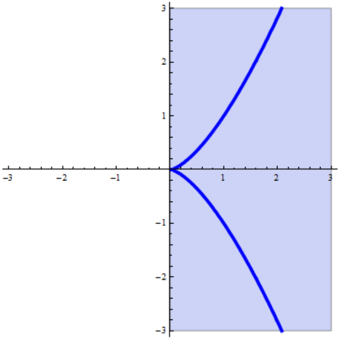
However we do not always have convergence.

Bad Example

Consider the ideal $I = \langle y^2 - x^3 \rangle$.



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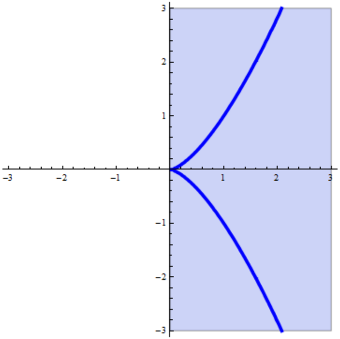

It is not hard to see that no linear polynomial is sos modulo this ideal, hence $\mathrm{TH}_k(I) = \mathbb{R}^2$ for all k.

Singularities and Convergence

Definition

For $x \in \mathcal{V}_{\mathbb{R}}(I)$, the tangent space of x, $T_x(I)$ is the affine space passing through x and perpendicular to ∇g for all $g \in \sqrt[\mathbb{R}]{I}$.

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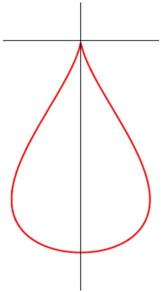
Proposition (G.,Netzer)

If I has a convex-singularity then, for all k,

$$\mathrm{TH}_k(I) \neq \mathrm{cl}(\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))).$$

Examples



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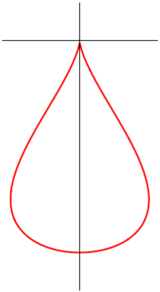

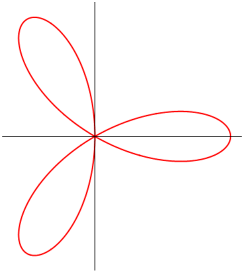

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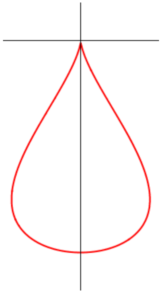

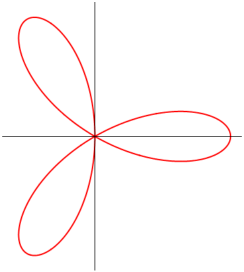

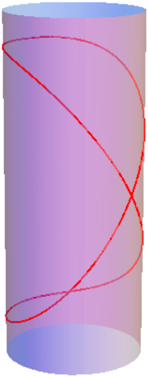

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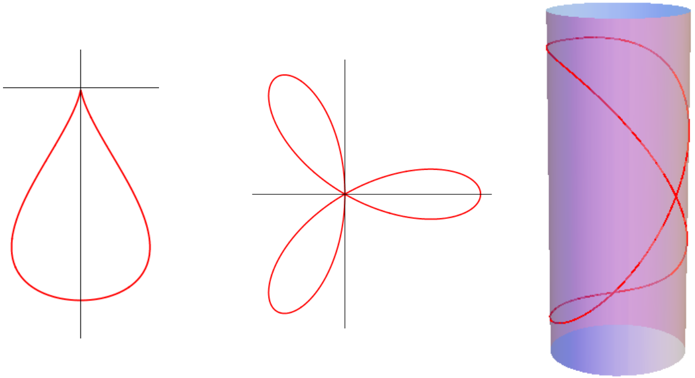

The first variety has a convex-singularity, but none of the other varieties have it.

Zero-dimensional varieties

In combinatorial optimization, zero-dimensional varieties ($0/1$-optimization) play an important role.

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Given a graph $G = ([n], E)$ find the maximum set $S \subseteq [n]$ such that no two points in S are connected with an edge.

This can be modeled by the ideal

$$I = \left\langle x_i^2 - x_i, x_j x_k \ : \ \forall i \in [n], \{j, k\} \in E \right\rangle$$

since $\mathcal{V}_{\mathbb{R}}(I)$ is the set of characteristic vectors of stable sets.

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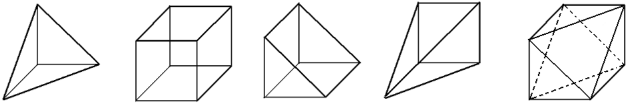
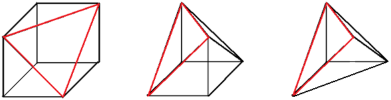
- For the stable set problem the ideal is TH_1-exact if and only if the graph is perfect. (Lovász)

Examples in $\mathbb{R}^3$

TH_1-exact



Not TH_1-exact



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To do as in the algebraic set case we have to find an algebraic certificate of nonnegativity over S.

Nonnegativity over semialgebraic sets

A classic way of certifying nonnegativity of a polynomial p over S is to provide a representation

$$p(x) = \sigma_0(x) + \sum_{i=1}^{k} \sigma_i(x) g_i(x)$$

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We will denote by Σ_d^S the set of all polynomials that have such a representation with $\deg(\sigma_i g_i) \leq 2d$ for all i.

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Lasserre Bodies

$$\mathcal{L}_d(S) = \bigcap_{\ell \text{ linear }, \ell \in \Sigma_d^S} \{x \in \mathbb{R}^n : \ell(x) \geq 0\}$$

which we call the d-th Lasserre Relaxation of $\text{conv}(S)$.

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- If S is Archimedean (i.e., has an algebraic certificate of compactness) then we have asymptotic convergence of the hierarchy $\mathcal{L}_d(S)$. (Putinar)
- We are interested in finding when does finite convergence not hold.

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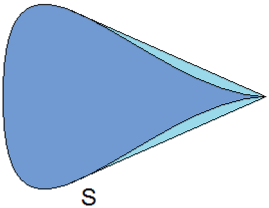

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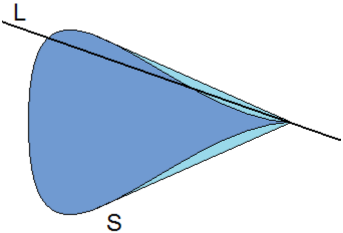

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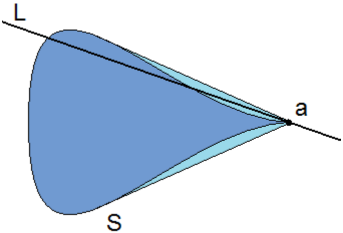

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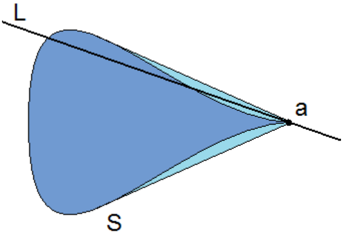
If for all g_i *s.t.* $g_i(a) = 0$ *we have* $\bigtriangledown g_i(a) \perp L$ *then, for all* d*, we have* $\mathcal{L}_d(S) \neq \text{cl}(\text{conv}(S))$.



Singularities

Corollary

If S has non-empty interior and there exists a point $a \in S$ that is on the boundary of $\operatorname{conv}(S)$ *s.t. all g_i verifying $g_i(a) = 0$ are singular at a, then we have $\mathcal{L}_d(S) \neq \operatorname{cl}(\operatorname{conv}(S))$ for all d.*

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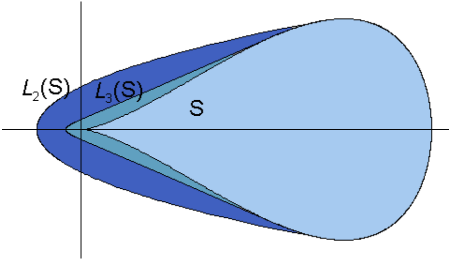

$$g(x, y) = -x^4 + x^3 - y^2$$

Non-exposed faces

Corollary (Netzer-Plaumann-Schweighofer)

Suppose S is convex and has non-empty interior. If S has a non-exposed face then $\mathcal{L}_d(S) \neq \mathrm{cl}(\mathrm{conv}(S))$ for all d.

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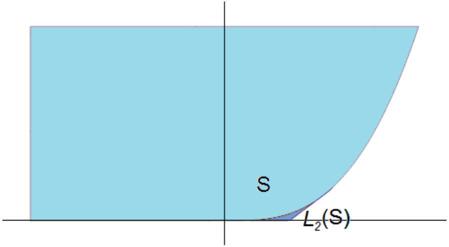

$$g_1(x,y) = y - x^3,\ g_2(x,y) = y,\ g_3(x,y) = x+1,\ g_4(x,y) = 1-y$$

The End

Thank You